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Science and Society*

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I.

IT is a high honor to be invited to address this gathering of men who search for the kind of knowledge on which our present industrial civilization is based. When we analyze the economic and social structure of the world in which we live, we find that it rests upon a vast foundation of scientific discovery. The steam engine, the development of electric power and its application, the internal combustion engine, the fundamental steps in chemistry and metallurgy which made large-scale steel manufacturing possible, were vital technological inventions. They brought in their train such social institutions as the factory, the large city, the trade union, advertising, instalment selling, and mass production.

Underneath these technological inventions there lies a painfully accumulated store of fundamental knowledge concerning the properties of matter. That is the domain of the physicist. Without the method of exact scientific thought, and the physical laws discovered and developed by the great physicists—Aristotle, Bacon, Newton, Faraday, Maxwell, and many others who lived before the present century—there would have been none of these inventions, or they would have been evolved at a much slower rate, and certainly not to the same degree of perfection.

The industrial revolution of the nineteenth century—the "machine age"—had its inception in the work of the physicist. Since the earliest days, it was he who led the way in the search for undiscovered facts and principles of nature. Upon these, piece by piece, we have reared our present economic structure.

David Sarnoff

Few of us fully realize the long, continuous chain of evolution that has led from the purely scientific and theoretical work of men like Newton and Maxwell to the technology which we see all about us. Few men who are not professional physicists realize the implications for the future of humanity that lie in the theoretical and experimental work being done by the leaders in physical research

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today. The relationship of the physicist to the far-reaching social changes which follow his discoveries is one of great importance to all who are concerned with human progress, whether they be scientists, business men, or government officials.

II.

One could scarcely find a better example of the evolution of a scientific idea—starting in pure mathematical theory and achieving a practical result affecting the daily lives of all civilized people—than the one provided by the industry with which I happen to be associated, that of radio. It is one of the most recent fields of applied physics. But it is also one in which the debt to pure physics can be most clearly traced.

Radio had its origin in the purely theoretical reasoning of James Clerk Maxwell, a professor at King's College, London, when in 1865 he advanced reasons for the existence of electromagnetic waves. Twenty-two years later, Professor Heinrich Hertz at Bonn University, inspired by Maxwell's theoretical work, proved by experiment the actual existence of these waves and their ability to travel through space. Following his publications other university scientists began experimenting with electric waves. Work went on in the laboratories, and crude transmitting and receiving arrangements were devised. At this point, Marconi, in Italy, became interested. In 1895 he invented the elevated radiator, or antenna as we now call it. By means of improved transmitting and receiving equipment, a ground connection, and a telegraph key, he put together the first commercially successful method of transmitting electric waves through the air over considerable distances, and gave to the world a practical system of wireless telegraphy. From that point, development of the new invention progressed rapidly.

The social results of radio have been far-reaching, and the end is not yet in sight. Dr. William F. Ogburn, professor of sociology at the University of Chicago, not long ago compiled a list of 150 social effects directly traceable to radio. From Maxwell's theory of electromagnetic waves to 150 social results, the links in the chain are continuous.

Since the beginning of the twentieth century, many men in your profession have explored the physical phenomena which have given us the basis of the greatest developments in radio. You are all aware of the tremendously important part the vacuum tube plays in our industry. This device is the direct result of early work by physicists in the study of electron emission from heated filaments. Their work stimulated further study in the application of high vacuum technique to the development of modern tubes. Many men who became famous in other fields of physics contributed substantially to the early growth of radio.

Today the radio industry includes not only countless radio devices as such, but sound-amplifying systems, sound-motion-picture machines, photoelectric apparatus, and all sorts of vacuum tube applications. Every branch of physics has contributed to the creation of these things: mechanics, heat, light, sound, atomic physics, electricity and magnetism. Mechanics is applied to obtain uniformity of movement in sound-motion-picture machines and in phonographs; heat, in the development of high power vacuum tubes and loudspeakers; light, in the optical systems of photographic sound recording and reproducing equipment; sound, in loudspeakers and other vibrating mechanisms, and in the design of broadcasting studios and auditoriums; atomic physics, in vacuum and cathode-ray tubes; and electricity and magnetism, in all circuit design, as well as in construction of all magnetically operated devices.

In our utilization of particles of the atomic system we stand today where the early astronomers stood in their exploration of the heavens. They studied the planets and the stars which seemed to be the heavenly bodies nearest them. In our knowledge of the atom,. we first discovered and utilized the negative electron—the outermost and most easily accessible structural element, and the one which, in a sense, is nearest to us. Some day we shall know more about and doubtless utilize some of the other elementary nuclear particles which have been discovered in recent years—protons, neutrons, positrons, deutrons, and their various combina-

tions. These new discoveries in turn may give us new vacuum tubes, new sources of power, new modes of travel and communication, new manufacturing processes, new forms of illumination, new cures of dreaded diseases, new highways to health.

Even as the astronomers penetrate farther and farther into the depths of space, using ever more powerful telescopes, so does the physicist, with his bombardment apparatus, penetrate deeper and deeper into the atom. The fact that one of his most powerful tools, the cyclotron, utilizes an ultra-short wave radio transmitter as one of its basic elements, illustrates the relationship which exists among all the physical sciences and their applications.

III.

Radio—which grew from the seed planted by physicists to the point where it affects the life of nations—has "arrived," but only at an early station on its journey. We are just beginning to enter, in any practical way, the fascinating domain of ultra-high frequencies—in which radio sight will be added to radio sound.

Short wave transmission of pictures and printed or written material has been an accomplished fact for several years. It is now in daily service between Europe and America. The broadcasting of a facsimile newspaper into every business office and home—in half-hourly instalments if desired—is perfectly feasible. The establishment of such a service is now an economic rather than a technical problem.

The new art of television is also making progress. For the past ten months we have been transmitting experimental television programs from the Empire State tower in New York City to receivers in the hands of engineers at observation points throughout the metropolitan area. We believe that acceptable standards of picture definition, to which transmitters and receivers will be synchronized, have been reached.

No field of applied science leans more heavily on all branches of physics than does television. This is particularly true with respect to the recent work in atomic physics. Unexplained electronic phenomena occur in "iconoscopes," the devices which convert light into electrical currents, and in the cathode-ray tubes, or "kinescopes," which convert electricity back into light. Here is an absorbing and fruitful field of research for the modern physicist.

The major obstacles to the public introduction of television are no longer in the field of research and engineering. They lie in a new domain. Television now demands the creation of a new art form, allied with, yet distinctive from, the arts of the stage, of the motion picture, and of sound broadcasting. It requires new talent, new techniques of writing, direction, and studio control. It must set in motion an ascending spiral whereby good programs create a demand for receiving sets, thus creating a growing audience, which in turn will make possible better programs. Television must build networks, and justify an economic base capable of supporting an expensive program service. These are some of the problems of television, solution of which will one day make it a major industry. They are the kind of problems that sprout from a seed planted in the soil of pure physics.

An evolutionary sequence of thought, experiment, production, and use, similar to that which constitutes the history of radio, might be traced in almost every product or service at the command of mankind today. The food we eat, the clothes we wear, the roofs that shelter us—as well as our services of communication, transport, health, education, and recreation—all bear witness to the impact of scientific thought and processes upon human activities that are older than recorded history.

IV.

Yet that impact had scarcely begun to make itself felt a century ago. Prior to the 19th century, technological advances were largely the result of ingenuity or guesswork or trial-and-error. The brave soul who dared to dream dreams and see visions in the realms of pure science was regarded as a sort of witch-doctor—possibly harmless, or perhaps a suitable subject for an inquisition and burning at the stake.

I congratulate you, gentlemen, upon having been born to your distinguished calling in a more enlightened age. Society today honors you and acknowledges its debt of gratitude to you. Our radio, the newspapers and the magazines

reflect a new public taste for scientific subjects. They dramatize your discoveries. Every man who reads or listens can acquire a smattering of scientific knowledge, and converse about electrons and relativity and light-years. A light-year is something like a billion dollars: we have learned to talk about it even if we can't quite comprehend it.

The scientific approach to the solution of human problems, the cumulative effect of our ever-increasing store of knowledge, the speeding-up of technological improvements through the coordination of discoveries in diverse fields—these are assets of such recent origin that the ink with which mankind has entered them upon its balance sheet is scarcely dry.

V.

It is one thing, however, to have assets, and another to know how to use them. The inductive processes of thought by which science lays brick on brick—testing each as it goes along to make eventually an impregnable structure—are not the processes we use in our attempt to construct a better social order. Whereas the scientific world glorifies truth, the social world still operates largely with prejudice. Thus it comes about that we have assets which are products of the scientific type of mind, but we do not yet know how adequately to use those assets to make them pay dividends to humanity.

Humanity's balance sheet is all right. It shows resources that are priceless, and a net worth that is beyond computation. But the value of assets without yields is purely theoretical. The document we need to be concerned about is the human profit-and-loss statement.

The scientist and the sociologist have viewpoints which are perhaps too widely separated. The scientist is engaged in the pursuit of truth, the knowledge of which is the most valuable of our acquired assets. The sociologist is concerned with the ultimate effects of truth upon human behavior. To the scientist the discovery of truth is an end in itself. To the sociologist it is only a means to an end. The scientist is interested in our balance sheet. The sociologist is interested in our profit-and-loss statement.

In an industrial age, the difference between profit and loss for humanity is the difference between use and misuse of the products of industry.

In bygone days the principal products of the industrial anvil were simple plough-shares, but, as science has given us infinitely richer and more varied materials with which to work, our modern industrial output has become more and more complicated. Today the sociologist rightly may claim that many of the gifts of science and industry are in the nature of a two-edged sword. It is a sword which, like the *Nothung* of Siegfried, can be used to slay the dragons of ignorance, intolerance, and greed; but there is always a chance that it will turn out to be a weapon with which civilization may destroy itself.

One does not have to go far to find illustrations of the blessings and dangers that go side by side in the discoveries of science. The chemical that safeguards the work of the surgeon can poison the city's water supply. The airplane, that speeds transportation and commerce, can drop bombs from the air to blow women and children to atoms.

There are other dangers, less obvious and more subtle. Radio, for example, can be used for propaganda and regimentation, as well as for education and entertainment. Science laid the same gift of radio at the feet of society in Europe as it did in America. It is true that in the United States there is room for improvement in some of the programs broadcast on the air, and that we still have to learn how to derive the greatest social benefit from radio. But no one raised in the tradition of liberty and democracy can doubt that our use of it is in the direction of social betterment, and that in certain parts of Europe, where radio has been commandeered by the forces of regimentation, its misuse points toward social degradation.

The new art of television has similar potentialities to build up or tear down social values. Like sound broadcasting, it can make friendly neighbors of people who differ in race, creed, politics, and language; while at the same time it offers a powerful weapon to the war-maker, and a medium of propaganda for the autocrat.

Fortunately, man's deliberate abuses of the gifts of science are the exception rather than the rule. Otherwise we would have to agree with the ancient and pessimistic philosopher who

asserted that, "*He who increases knowledge increases sorrow.*" Such a belief would warrant converting our universities into trade schools, and our doctors of science into expert mechanics.

On the other hand, it is too optimistic to assume that the mere translation of a scientific discovery into a usable commodity or instrument always advances civilization; that just because humanity can travel faster, communicate more freely, cook, wash, iron, and gather ice cubes with less effort than ever before, it has reached the all-time peak of civilization. Giving a man a hoe or a microscope does not make him a farmer or a scientist, and giving him a radio or an automobile does not make him civilized. It is the use that society makes of the products of science and industry that determines whether civilization is advanced or retarded.

VI.

Civilization depends for its advance upon our expanding knowledge of the social as well as the physical sciences, for society, no matter how benevolent its intentions, cannot solve its problems by intuition or rule-of-thumb. It must develop its own standards and technique. Yet it is only natural that the growth of social science should lag behind that of physical science, for the social scientist is dealing with human rather than inanimate materials; with moral rather than mathematical equations; with experiments which yield their final results, not in the course of days or years, but only in generations and centuries.

Experimentation in any field is an undertaking of the pioneering spirit—never that of the reactionary and the standpatter. The advance of social science, no less than that of physical science, calls for the creative imagination of a Newton and a Maxwell, an Edison and a Marconi. Obsolescence is a factor in social as well as in industrial machines. In society as in industry, to stand still is to go backward.

The social experimenter is important to society. But if he is ruled by his emotions, and abandons the spirit of research for that of crusading, there is danger for society. The spirit of the social crusader is not always the spirit which seeks for truth. To attempt to cure the ills of society with untested remedies is not the method of science.

The social experimenter must be one who approaches the problems of society in the clear light of an unbiased mind. He must collect and analyze facts, seek to fathom causes and establish principles. He must always be willing to reconsider those principles in the light of their results.

It is, of course, impossible to draw a complete analogy between the world of science and society itself. The engineer deals with things that have no will of their own. His is the will, which can command them to take a certain shape and serve a certain purpose. He who would be an engineer of society must deal with human beings, who are flesh and blood, emotion and prejudice, dreams and aspiration. They each have a will, and in a democratic society people must be persuaded and not coerced. But the scientific spirit can direct the means of their persuasion. A surgeon does not try to persuade a patient to undergo an operation until he has made a careful diagnosis, and can convince the patient that the operation will help him. He appeals not to his prejudices or to his fears, but to his sense.

If the cause of civilization is to be advanced on a permanent basis, we must learn to follow a procedure similar to that of the experienced surgeon. We must not be afraid to operate; but it is vitally important to know when to operate and when to advocate a less spectacular and safer treatment. One decision may require as much courage as the other. We must investigate and verify social and economic facts; analyze them in order to arrive at logical conclusions; and obtain widespread acceptance of those conclusions through popular education and debate.

All these are slow, laborious processes, and may prove irksome in a fast-moving age. But they are necessary if we are to preserve a free society. The enlightened citizen accepts the fact that the momentum of society, like the momentum of physical bodies, is the product of mass times velocity. The velocity of the few often has to wait upon the inertia of the many, and it is only by overcoming that inertia that genuine social progress among a free people is achieved.

VII.

Industry today is following the vanguard of science into new and infinite realms of knowledge. It would be a rash astronomer who said that he had calculated the outermost limits of space, beyond which there is nothing. It would be a rash physicist who claimed that he had dissected the atom into its ultimate, indivisible fragments. Science and knowledge have no boundaries.

So it would be a rash economist who predicted any limit to the tangible results of scientific thought in the form of new goods and services placed at the disposal of mankind. In fact, it is only by a constant development of new goods and services that we may expect to re-engage the man-power released by technological improvements in established industries. The market for every new commodity eventually reaches a saturation point and becomes primarily a replacement market, so that a more efficient technology reduces the number of workers needed in that field.

But science is simultaneously creating new employment, both by the modernization of established industries and by the creation of new ones. In our own generation we have seen the automobile, the airplane, the motion picture, and the radio provide totally fresh fields of activity for millions of men and women. Many of our older industries have engaged scientists, with notable success, to develop new and remodel old products to meet the needs of a modern era.

The industry which has not learned how to employ scientists to make it new, and keep it new, is doomed. Few industries are so stagnant as not to be aware of this; but there are some so conservative that the scientist is called upon to turn salesman and show them how modern science can rejuvenate them to meet present-day realities and survive.

In all respects, I hope we can bring about a closer understanding and cooperation between you, who seek new truths in the universities, and leaders of industry, who seek to make the truth you discover serve society. Not only research staffs but industrial managers should at all times be kept informed of your new discoveries. With such knowledge, promptly obtained, I am certain we can shorten the time-gap which now separates technological unemployment and useful re-employment.

Any measure of unemployment relief obtained by placing a check-rein upon technology, or by arbitrarily hampering men's efficiency, is unsound, uneconomic, and cannot endure.

VIII.

The problems created by technical science must be solved by increasing and applying our knowledge of social science. Society cannot expect to see every detail of its future plotted in advance, any more than the scientist in his laboratory knows beforehand where each new experiment will lead him. But a study of the social strains and stresses in the world today points toward a clearly defined objective which should be the immediate goal of social science. That goal should be to achieve economic justice, peace and prosperity in a free democracy.

The autocracies of Europe have come into power by exploiting fears of insecurity and by promising impossible Utopias. Fear is an enemy of progressive development, whether of individuals or of nations: fear of economic and social forces too powerful to be struggled against; fear of insecurity, of injustice, of unpredictable economic changes.

The American pioneer in the pre-machine age did not know this kind of fear. His enemies were the forces of Nature from which he could wrest a living with the strength of his two hands.

That strength is no longer enough. The individual is no longer self-sufficient. The fears of today must be banished by the organized effort of an enlightened society, capable of bringing into equilibrium the economic forces that huge populations and the industrial age have set in motion.

Because a collective effort is called for, however, it is a false and dangerous assumption to think that the freedom of the individual must be turned over to the state. The governments which have demanded such a price have not been able to deliver the economic security they promised in exchange. But even if they could make good their promises, the belief that any material gain is worth the loss of individual liberty denies the

spirit and tradition of America, and ignores the bitter lessons of history.

IX.

Freedom of the individual is essential to the full expression of his creative faculties—whether in art, in science, or in technology. Only out of free minds has modern civilization been created; only out of free minds can it be advanced.

To reach our goal of economic and social freedom, it is not sufficient merely to have an idealistic viewpoint. Idealism is only wishful thinking until it is given shape and direction by fact-finding and logic. We must learn to apply science to the business of government and to the government of business.

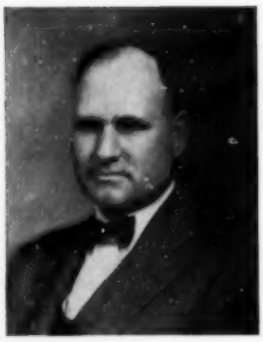
At a time when the people of many countries are being encouraged by their leaders to believe that salvation lies in autarchy and self-sufficiency, it is well to remember that the minds of men in all nations have collaborated across all borders to advance the service of science to society. Without such collaboration, progress will be slowed up, and may even stop. Regardless of how the human emotions may operate, the human mind knows no national boundaries.

Science and Society depend upon each other for their welfare and advance. Science teaches a basic lesson—that knowledge of the truth, without fear or prejudice, is indispensable to progress. Where that lesson is rejected, and scientists are ordered to shape their reasoning to fit the purposes of an autocratic government, science goes backward instead of forward. The tragic position of a scientist in a regimented society was recently revealed by a professor in one of Europe's oldest universities, when he made this public statement: "*We do not know or recognize truth for truth's sake or science for the sake of science.*"

Under such a doctrine, Science and Society can only serve each other badly. Democracy believes in a different philosophy, written in words of wisdom nearly two thousand years ago: "*Ye shall know the truth, and the truth shall make you free.*"

As a result of all my reading and observation it seemed to me that improvement in the standard of living of the human race could come about in the future only by reason of new discoveries and inventions, just as, in the past, the steam engine and other inventions had been responsible for many improvements in the standard of living enjoyed by the average man today. It was these things, and not governmental or political action, that had increased production, lowered costs, raised wages, elevated the standard of living and so had brought about a greater participation of the human race in these benefits.

—ANDREW W. MELLON—*Address at the Dedication of the New Building of Mellon Institute, Pittsburgh, May 6, 1937*



Alloys for Tools and Wear Resistance*

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SOME twenty years ago there appeared in one of the French periodicals a statement to the effect that when metals are subjected to great hydrostatic pressure the hardness is substantially increased. That didn't seem to me to be a logical thing. I didn't see how a metal could be hardened without being plastically deformed—that is, a metal which was not susceptible to heat treatment. Dr. Bridgman collaborated with me at the time in some experiments, and we demonstrated quite conclusively that when the pressure is truly hydrostatic there is no plastic deformation and there is no change in hardness that could be measured.

Dr. Bridgman's diagnosis of the experiments made in France was that the fluid which had been used was one which itself solidified under high pressure, and in that way the supposed hydrostatic pressure was nonhydrostatic and made possible a certain amount of plastic deformation which, in turn, gave rise to the hardening.

In connection with tools, I want to say in passing that when one gives a title, "Alloys for Tools and Wear Resistance," he is not necessarily limited to the hard materials. Many hammers used industrially are made of lead. The old time hammers, of course, were hard, and most of the present day ones are relatively hard.

There are many uses for soft hammers. One of them is in the molding art, and especially in connection with permanent molds. These molds must be hammered from time to time to separate one part from another, and it is essential that the mold part remain unimpaired. A lead or aluminum hammer may be used for that purpose. Of course, the hammers easily deform and quickly wear out, but their replacement is not expensive. It is well to remember that tool materials are not necessarily hard.

However, it is my intention to discuss in a broad way certain of the harder tool materials. We have all read of the profound effect that tools have had on the progress of civilization. It has been stated that the progress of peoples can be traced through their development of tools and tool materials. That is quite obvious from historical studies.

The application of tools to our modern industrial development, however, is not quite so obvious. What is happening in the industrial world is the transfer of hand tools to machines. Consider, for example, the anvil, the hammer, shears, punches, knives, saws, chisels, tongs and the like. These came into existence originally as hand tools. For centuries, much of the progress of the civilized world depended on the development and use of these as hand tools.

Our oldest tool material of any great moment is carbon steel. The carbon steel tool is still used as a typical tool material in tremendous quantities. It is used, however, for machine parts in even greater quantities. Take, for example, lathe chucks. These involve some combination of the tongs and the vise, being a mechanical

* Presented at the Symposium on Metals, Cambridge, Massachusetts, January 28–30, 1937.

grip to serve the same purpose as the hand grip.

The use of tool steels for such purposes is very much greater than it is for tools themselves. We can visualize literally thousands of types of industrial machines today whose function is essentially that of holding tools which simulate the old time hand tool and even the old time hand operation. In one machine, many combinations such as the grip, the file or its equivalent, the saw, the shear, and the punch, operate simultaneously or in quick succession and produce parts all the way from hundreds per minute, down.

The evolution, then, of a tool material is from its use as a tool, and usually in the old days as a hand tool, to a machine part. Let us consider one use of what used to be a typical tool steel for ball bearings and roller bearings. The steels used for these parts are tool steels. The tonnage of ball bearings and roller bearings today is greater than that of the same materials used for the simple tools. Other examples will occur to you, illustrating the tremendous use of these old time tool materials in machines and as part of the equipment of our modern industrial civilization.

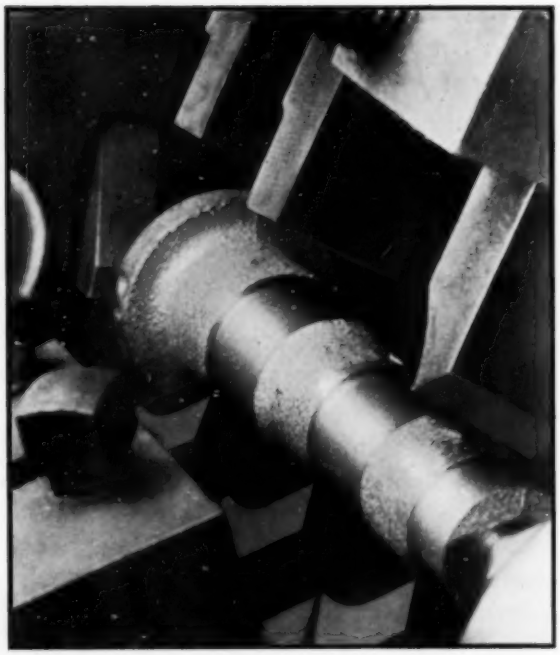
When high speed steel came into use in the early part of this century, many people thought it would replace carbon steel in tools. It did replace many carbon steel tools, but there are more carbon steel tools made today than there were in the first part of the century. What high speed steel has done is to make possible a greater machine age, and it has taken up part of the growth of the tool industry, but it has not decreased the output of carbon steels.

High speed steel first came into use as a tool material, and now it has many other uses. It is destined to have still more uses outside of the typical tool field. The same is true of stellite, which is essentially an alloy of cobalt, chromium and tungsten. When it is cast it has full hardness. It is not machineable, like high speed steel, because it has no soft state. High speed steel has the virtue of being soft enough in one state to be machined. After it is machined it can be hardened to any desired degree up to about 65 Rockwell "C." Stellite must be cast to shape, or nearly to shape, and formed finally by grinding. It is true that it can be machined to an extent with diamond-pointed tools or with some of the newer cemented carbide tools, but the forming of stellite tools by machining is not a large factor at the present time.

Stellite first came into use as a tool material, and now it is being used in greater quantities for machine parts, not only for hard surface facing, but also as cast stellite parts themselves, for wear resistance.

The newer tool materials are the cemented carbides, representing a new art. The utilization of powder metallurgy in the production of these materials represents an advanced step and a new step in the tool art. This industry at the present time is well launched the world over. The material handles more like diamond than like a metal at the present time. On the average, in the United States, a pound of the material is divided into ninety pieces before it goes into use. Even so, in the year 1936 upwards of fifty tons of material were consumed throughout the world. Even the cemented carbide has already started in the direction of use for machine parts as distinguished from the tools themselves. I have in mind one machine which uses 25,000 cemented carbide wearing points. Within a period of two or three months there was supplied the material to equip this particular machine and also one die for drawing steel tubing. The die in one piece weighed five and a half pounds. The 25,000 pieces for the one machine weighed five and a half pounds. The application to the machine was equivalent to making 25,000 small tools, because each piece of cemented carbide was brazed or welded onto the end of a small steel shank.

The above is mentioned only as an example. The cemented carbides are being used successfully in many wear resistance applications in many types of machines, as, for example, in lamp-making machinery. One of the parts in one of the machines is a clamp which grips a piece of tungsten wire and holds it tightly while it pushes one end into hot glass: that is for the purpose of making the filament supports. Until the cemented carbides were available, these clamps had to be changed sometimes three times a day. Before a change, as a rule, there would be some defective mounts. With cemented carbide



Turning and facing a cast iron textile machinery part with carbide tools at 1170 r.p.m. Production per hour: 150 pieces. Number of pieces between tool grinds: 12,000 to 1400.

clamps the machines can run along for weeks without any attention.

Another example is in the application of cemented carbides to cutters and anvils for flattening wire. The difference in life is at least three or four hundred to one over the best materials available prior to the use of cemented carbides.

The advantage, of course, is not only in the longer life itself and in the fewer defective parts. Some of the advantage derives from lessening the attention required for the operation of the machine. If one has to change a part two or three times daily, the machine needs almost constant watching during operation. If the part has to be changed only once in six months, the operator need give little attention to that particular part.

The cemented carbides are used for valves and valve seats. The control valves in certain ice-making machines are made of cemented carbide —not only the seats, but the needle valves as well. There are other types of valves which are relatively inaccessible, such as those in mine pumps and pumps which must handle not only a clear liquid but slimes and materials containing grit. It has been found that the lives of valves made of cemented carbide under such circumstances may be hundreds of times those made of the ordinary valve materials.

Although the cemented carbide industry is young, it has within a few years changed from use as a tool material solely to application in machine construction. Hence it is undergoing in the course of a few years the type of evolution through which the ordinary carbon tool steel took thousands of years to pass.

I wish to say something about the intermediate material, the alloy of iron, cobalt and chromium. By substituting iron for the chromium of stellite and changing the cobalt and tungsten content materially, an alloy can be made which is not hard when it is first cast. In fact, it can be made of a hardness which is machineable. This alloy can also be forged and rolled, and then it is susceptible to heat treatment. There is no carbon —or at least not enough to produce carbon hardening. The carbon content may be up to a quarter of a percent without changing the hardness much and without interfering with the heat treatment. After this alloy is forged and rolled, it can be heated to a high temperature, say 1250° centigrade, and cooled fairly rapidly, after which it is relatively soft and machineable. The hardness at that stage is about 45 Rockwell "C." It can then be reheated to 550 or 650° centigrade and the hardness can be brought up to any value desired between 45 and about 70 Rockwell "C."

The alloy has some very strange characteristics. In certain conditions it is so brittle that when dropped on the floor it will break. Also, if the high temperature treatment is not high enough, for instance 1150 instead of 1250°, the hardness can be developed with ease but the material has poor cutting properties. This is an unexplained phenomenon for some of the younger metallurgists and physicists to work on in the future. Why do we get the requisite hardness and yet poor cutting properties?

The alloy can be given good cutting qualities by suitable heat treatment, and it has some rather peculiar properties still not explainable when it is produced in the preferred state. It has a peculiar behavior in machining the austenitic steels. It machines Hadfield's manganese

steel better than anything that we have ever seen, the cemented carbides included. It machines the stainless steels better than any other tool material. Why that is I do not know, but I am offering it as one of the unexplained facts. In the course of time we shall know more about it.

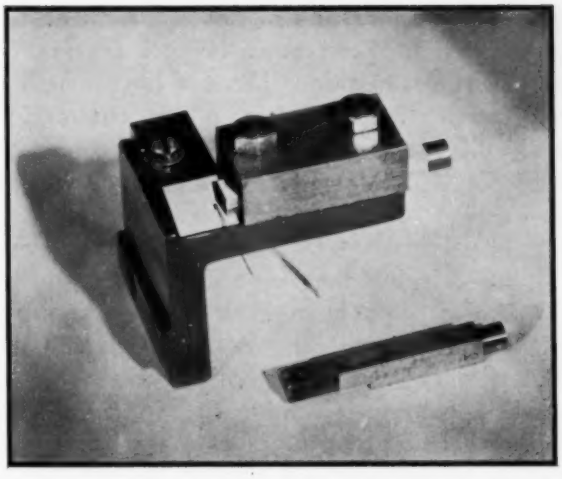
I may say, too, that we do not yet know why this material is so weak after a certain heat treatment. That is another unsolved problem, so far as we are concerned.

Even this material, new as it is, and used to only a very limited extent for tools at the present time, is already finding use outside the tool field. It is an even newer material than cemented carbide and yet, within two or three years, it is going through the same type of evolution which took thousands of years for the ordinary tool material.

Finally, I wish to mention some things about the carbide materials which may be of interest. Of all the foundation carbides, tungsten carbide is, so far, the best. Why that is we do not know. It seems to have what Dr. Hoyt stressed a number of years ago in connection with tool steels—"body." It is relatively insensitive to temperature changes, and has a toughness out of proportion to its strength as determined by transverse tests. It is a fact of experience that tungsten seems to be the best material. Tungsten is the foundation of nearly all the commercial grades of cemented carbide. The most important of these materials is straight tungsten carbide and cobalt, the cobalt ranging, according to the toughness desired, from about three percent to, say, twenty percent.

There are other materials, however, which have found definite places in the art. One is tantalum carbide, and the other is titanium carbide. Some material is made containing no tungsten carbide, but it is very small in poundage—that is, tantalum carbide cemented with nickel or cobalt, or both. Most of the tantalum carbide is used in conjunction with tungsten carbide. Cobalt is used as the cementing metal, although nickel can be used, and both nickel and cobalt in the same mixture can be used.

The tantalum materials do not crater in steel turning or steel cutting as badly as do the straight tungsten carbides. That phenomenon is not understood. If twenty percent, say, of tantalum carbide is added to tungsten carbide, the steel cutting qualities of the material are greatly enhanced.



Carbide shearing blade for shearing steel tag wire. Carbide tools shear 1,000,000 pieces between grinds as compared with 40,000 to 60,000 with high speed steel tools.

Titanium carbide acts in somewhat the same way, but different in magnitude. The materials which can be used at the highest temperatures are those which contain large amounts of titanium carbide. The cutting speeds attainable in steel are beyond the capacity of most existing lathes and machine tools at the present time. Soft steel can be cut up to fourteen hundred surface feet a minute. Very few commercial machines will stand that speed. Why the titanium behaves as it does is just as much of a mystery as that relating to tantalum carbide.

The surface of this field has barely been scratched. While these three materials are all used industrially and are important at the present time, that doesn't mean that the end has been reached in connection with these cemented tool materials.

There is one recent industrial development which I think is very important. Even though the tantalum carbide and titanium carbide materials will cut steel, in this country the art of cutting steel with the carbides has advanced slowly. In Europe about thirty-five percent of the cemented carbide is used for steel cutting. This is easy to explain. They have a large number of single tool machines. Conditions can be fixed for favorable operation with the carbides. Here, to a large extent, a machine must perform

many operations, some at low and some at high speeds. High speed steel is well adapted to handle such operations because it cuts well at low speeds. The carbides do not cut steel so well at low speeds. It has been a problem to work out combinations of tool material and tool design to cut at low speeds as well as at high speeds.

Within the last year some of these complicated machine set-ups have been operated entirely with cemented carbide tools. By selecting tool materials and angles—in other words, by designing tool materials for the work—and by using them at the proper speeds and feeds, steel jobs are now being run entirely with cemented carbides. During the past year much effort has been expended on this problem and much information has been forthcoming. Inasmuch as the steel cutting field is much larger than the cast iron or nonferrous field, I think this development has great potentiality.

From these remarks you might say that the cemented carbides are on the way to replace high speed steels. That is no more true than it was true in 1900 that high speed steel was destined to replace carbon steel. The cemented carbides will take up a proportion of the growth which we hope is on the way in this country. High speed steel, fifteen or twenty years from now, will be used in greater tonnage than it is today. In other words, it will be a bigger industry as the years go by, just as the carbon steel tool industry, or its equivalent, is a greater industry today than it ever was before.

I look forward to the growth—the rapid growth—of the cemented carbide industry and the growth of these newer alloys which, incidentally, will provide wonderful materials for structural application in connection with high temperature equipment, but I anticipate also an increased use of ordinary tool steel, stellite and high speed steel.

There need be no fear of science becoming the prostitute of commercialism, as some seem to think. On the contrary, the widespread and growing invasion of industry by scientists is a wholesome sign. The scientists of our industrial research laboratories are the same men who, twenty years or so ago, found opportunities for work only in our universities and schools, or perhaps in the medical profession, or in some relatively obscure scientific bureau of the government. Indeed, many of them have come to industry from these sources, where they were honored and revered for their records of able and unselfish service to mankind.

Industry is changing these men but little, if at all. The mere act of moving from one city to another, from one laboratory to another, from an old allegiance to a new one does not change human character. These are the men who have given ideals and altruism to our universities, and industry is now giving them opportunity for expression in a new field. It is asking but one thing of its scientists—to create, for the practical benefit of their fellow men.

—C. M. A. Stine

Electronic Structure of Alloys

By J. C. SLATER

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DURING the last five years, the electron theory of metals has come from a point where the mere explaining of electrical conductivity was something of a feat, to a stage where its interpretation of alloys should be a definite help to practical metallurgists. Further progress in the theory is likely to come more and more from cooperation between theoretical physicists and metallurgists, and the advance of physical metallurgy in the next few years is likely to be almost as dependent on the electron theory of metals as the advances of the last generation were dependent on thermodynamics and the theory of solutions. For the first time we can make a clear picture of what metallic solid solutions and intermetallic compounds really are, and as the theory progresses their properties are bound to be more and more clearly understood.

The ordinary concepts of chemical valence, of course, are not applicable to the metallic state. Alloys are compounds of variable composition, and the occurrence of intermetallic compounds of practically fixed composition is determined more by peculiarities of the crystal structure than by valence laws. The forces between atoms in a metal, then, are not of either of the two familiar chemical types. They are not electrostatic forces between ions of opposite sign, as in the ionic crystals of inorganic chemistry; neither are they homopolar valence forces, as found in the familiar inorganic gaseous molecules, and in organic compounds. The first thing which the modern electron theory has done is to provide us with a definite picture of the nature of the forces which actually operate within a metal. In the last analysis, of course, the forces are electrical; the quantum theory has not changed the fundamental fact that all interatomic and intermolecular forces of significance in the structure of matter are electrical, or in some minor cases magnetic. An atom of an electropositive element, or metal, contains several loosely bound electrons, surrounding a much more stable positive core, consisting of the positive nucleus and a number of negative electrons. The loosely bound electrons are the ones which can be removed when the atom in question enters an ionic crystal. But in a metallic lattice these electrons are not removed, for there is no electronegative element to attract them. They merely become free to wander through the crystal, thereby carrying electric current. The picture of a metal, then, is roughly a sea of free electrons, in which are floating enough positively charged ions to make the whole thing electrically neutral. Different metals vary greatly in the fraction of the volume taken up by the positive ions. In an alkali metal, like sodium, the fraction may be only ten percent, the other ninety percent being taken up by quite free electrons, while in the transition groups of elements the ions may occupy fifty percent or more of the space. Now electrostatics alone is able to explain the cohesion of such a structure. In the metal, the volume of the whole structure is less than the sum of the volumes of the isolated atoms; that means that the free electrons on the whole are crowded closer to the positive ions than in the separated atoms, resulting in a decrease of energy as the crystal is formed, or an increase of

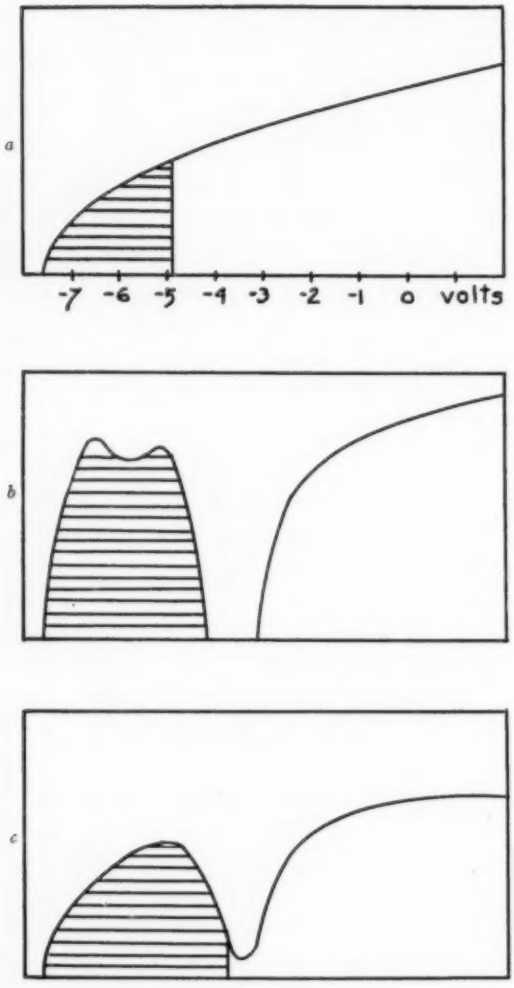

FIG. 1. Energy distribution of electrons in crystals. Abscissa, energy; ordinate, number of electrons per unit energy range. Shaded regions occupied by electrons. (*a*) Free electrons in metal. (*b*) Separated bands in an insulator. (*c*) Partly separated bands in a poor conductor.

energy as it is broken up, which would show itself as the heat of sublimation of the metal.

This explanation shows that the energy of the metal does not depend in any striking way on small motions of the ions. In an organic molecule, for instance, or in an inorganic structure like a silicate, the valence forces act in definite directions, like the tetrahedral bonds of carbon and silicon, and any displacement which changes interatomic bonds, either in direction or in interatomic distance, meets with large resistance. In a metal, on the other hand, an individual ion does not greatly care just how many neighbors it has, or in what directions they are located, with the result that it is easily deformed, and that the same metal often exists in several crystal structures, with different numbers of neighbors, or different coordination numbers, in the various phases. Furthermore, the metals with small ions, like the alkalis, are softer and more easily deformed than those with large ions, like the ferrous metals. The size of the ions is not the only factor affecting ductility, however. It is a familiar fact that intermetallic compounds of definite composition, and some few metals, show a surprising brittleness as compared with most metals, and at the same time show unusually small electrical conductivity. Thus in the copper-zinc alloys, the gamma-phase, a peculiar crystal structure occurring only for a narrow range of compositions, is much more brittle than the other brasses, and a much poorer conductor. One of the most conspicuously brittle metals is bismuth, which at the same time is a poor conductor. Many other examples could be given. A more detailed study of the free electrons gives the interpretation of these less ductile and metallic structures.

To understand these points, we must consider the kinetic energy of the free electrons. One might suppose at first sight that at low temperatures the free electrons of a metal would have no kinetic energy, and that at higher temperatures the mean kinetic energy would be $3/2\ kT$, as with a perfect gas. The situation is quite different, however, on account of the principle called Pauli's exclusion principle, whose counterpart in statistical mechanics is the Fermi statistics. According to this principle, no two electrons in the metal can have the same kinetic energy, but instead the electrons must be spaced out in energy according to a definite law. The result is that while the slowest free electron has no kinetic energy, it is the only one of this sort. There is then one with a small energy, one with a larger value, and so on, until the fastest of the free electrons has quite a large kinetic energy, of a magnitude equal to what it would get by falling through a difference of potential of several volts. This holds even at the absolute zero; increase of temperature, in fact, makes very little change in the energy distribution, so that the free electrons contribute very little to the specific heat. The law of spacing of the energies is simple for free electrons: the number of electrons

per unit energy range is proportional to the square root of the kinetic energy. This is shown in Fig. 1a, where the abscissa is the total energy of the electron, the ordinate the number of electrons, and where the energies actually occupied by electrons are shaded. We also show the zero of energy, drawn to be correct for sodium. All the electrons, as we can see, have negative energy. The total energy is the potential energy, which is negative, on account of the attractive forces between the positive ions and electrons which we have already mentioned, plus the kinetic energy, and to get the total energy of the metal we should have to take the average energy, or the center of gravity of the blackened part of the figure. This corresponds to the heat of sublimation of sodium, and the theoretical calculations check experiment fairly accurately.

Now the alkali metals are almost the only ones where the electrons act as if they were really free. The more difficult an electron is to detach from an atom, the less it acts like a free electron in the metal. An electron fairly tightly bound to an atom can only occasionally jump over to a neighboring atom, with the result that its average kinetic energy is very small. As a result of this, the energy distribution curves generally are higher, meaning that more electrons can be crowded into a given energy range, so that the mean kinetic energy is lower. There is also another change: the distribution curves are no longer smooth and regular, but show a structure such as is shown in Figs. 1b and 1c. These show two cases: first, in 1b, an extreme one in which the energies are separated into distinct bands; second, in 1c, an intermediate case where there is almost complete separation. In the first case, if one band is completely filled, the other completely empty, the crystal is an insulator. The reason is that an electric field in a conductor accelerates the electrons, giving them larger kinetic energy, in the process of building up an electric current. But in a completely filled band, there are no states of higher energy to which these electrons could go by the action of an electric field, so that no current can be set up. Such a pattern, in other words, is characteristic of an insulator; that is, in general, of a chemical compound held together by ionic or homopolar valences. The intermediate case, where the bands are almost completely separated, corresponds to a poor conductor, which at the same time approaches an intermetallic compound in properties. It is this intermediate case which we have mentioned above, as resulting in structures of low conductivity and low ductility, sort of a compromise between a metal and a chemical compound, of which two examples are the gamma-phase of brass, and bismuth, both previously mentioned.

The particular points which I wish to bring out in the present paper, however, do not relate to problems of poor conductors. I wish instead to see how this theory of energy bands applies to the iron group of elements, particularly to the ferromagnetic elements, iron, cobalt, and nickel, and to their alloys. Fig. 2 shows the energy distribution curve, which applies schematically for this whole series of elements. The typical feature is that there are two distinct bands of electrons, overlying each other in energy. The low, broad band, much like the distribution curve for free electrons, though on this scale it looks much lower, comes from the few easily removed electrons of the atom, the 4*s* electrons in the language of atomic structure, of which most of the isolated atoms of the iron group possess either one or two. The sharp, high band comes from the next inner shell of electrons, the 3*d* shell, which begins to be filled up with scandium, which normally has one such electron, and which is completed with nickel, which normally has ten electrons, the number which this band can accommodate. In any one of these elements, the band is filled up to a definite

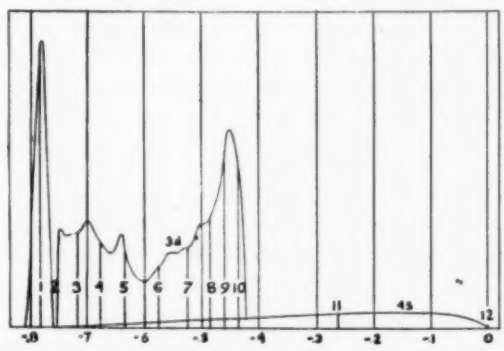

FIG. 2. Energy distribution of electrons in the iron group. Numbers 1–12 represent the energies to which the bands are filled in elements with 1–12 outer electrons.

point, such that the combined area of both 3*d* and 4*s* bands up to this energy is enough to accommodate the total number of electrons. In general, this results in a fractional number of electrons in each band. Thus in nickel there are ten outer electrons, of which about 9.4 electrons per atom on the average are in the 3*d* band, 0.6 in the 4*s*; in cobalt, the next preceding element, there are 9, about 8.3 in the 3*d*, 0.7 in the 4*s*; in iron, there are 8, about 7.3 in the 3*d*, 0.7 in 4*s*; etc. These figures are not very accurate, but are approximately correct. Of course, they do not mean that atoms really have fractional numbers of electrons; they represent averages over all the atoms of the metal, each atom having a whole number of electrons. Now the 4*s* electrons are the ones which behave in a typically metallic way, forming a sea of charge in which the positive ions float. The 3*d* electrons, on the other hand, are attached rather tightly to the ions, as the high narrow form of the distribution curve shows, so that they are not very active in conduction. One might think that they would give rise to binding forces like chemical valence, but they are rather too far apart to have a very large effect of this sort, though they do have some effect on binding. As the 3*d* shell fills up, the first electrons entering it of course have the lowest possible energy, and later ones have higher energies. At first, then, the 3*d* electrons tend to lower the average energy of the electrons, resulting in a progressively tighter and tighter binding as more and more are added until finally they begin having such high energy that they tend to increase rather than decrease the average energy, and the binding gets weak again. This fits in with the fact that criteria such as heats of sublimation, compressibility, melting point, and other properties show the binding of the elements in the middle of the transition group to be greater than the binding at either end of the group.

The electrical conductivity of the elements results largely from the 4*s* electrons, the ones which act mostly like free electrons. Since the number of these is less on the average than unity, the conductivity is poorer than for alkali metals; other features in addition decrease the conductivity, which there is not time to discuss. And the 4*s* electrons are also largely responsible for the binding and for the lattice spacing, which is remarkably constant through this group of elements, though we have just seen that the 3*d* electrons have some effect here. The most characteristic property depending on the 3*d* electrons is ferromagnetism. Consider iron, for instance, which has about 7.3 electrons per atom in the 3*d* band, as against a possible 10. To discuss magnetism, it is essential to know that the individual electrons are magnets, and that it is their magnetism which results in ferromagnetism. Now in the presence of a magnetic field there is a principle which limits these electron magnets to two orientations only: their magnetic moments can point parallel to, or opposite to, the field, but not at any other angle. Furthermore, the band of 3*d* electrons, holding 10 electrons as a maximum, is subdivided into two identical bands, one for electrons of each orientation, each holding 5 electrons. Let us now consider two extreme cases of magnetism of our iron crystal. First, it could be completely unmagnetized, even in its smallest elements. Then there will be 3.65 electrons per atom of each orientation in the 3*d* shell. On the other hand, consider the completely magnetized state. Then the best we can do is to have 5 electrons per atom parallel to the field, for that fills up the band, leaving 2.3 electrons pointing opposite to the field. The net magnetic moment is then $5-2.3=2.7$ electronic moments per atom. A similar calculation gives 1.7 electrons per atom for cobalt, 0.6 for nickel, and shows that copper, the next element beyond nickel, for which the 3*d* shell is completely filled, could not possibly be ferromagnetic. Now when we convert into practical units, these net mag-

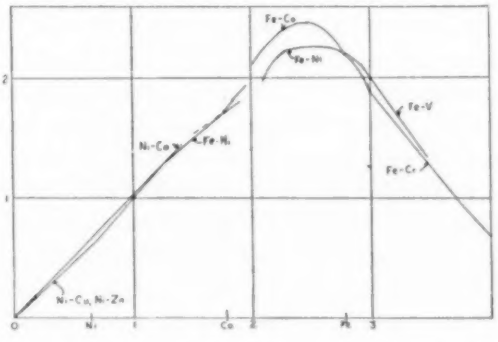

FIG. 3. Saturation magnetization, in electronic units, as function of number of vacant spaces in shell of 3*d* electrons, for binary alloys Ni-Cu, Ni-Zn, Ni-Co, Fe-Ni, Fe-Co, Fe-V, Fe-Cr.

netic moments actually agree with the observed values of the saturation magnetization for these three atoms. If we look back at the figures, we see that the general rule is that the saturation magnetic moment equals the number of empty spaces in the 3*d* shell, times the moment for one electron per atom. Of course, for the elements before iron, which are not ferromagnetic, this rule cannot be expected to hold.

Fig. 3 shows how well this rule regarding saturation magnetization holds for the whole series of ferromagnetic binary alloys of iron, cobalt, and nickel, and some neighboring elements. Here we plot the magnetic moment (in units of the electronic moment) as a function of the number of empty spaces in the 3*d* shell. The significant feature of this plot is the way we have included alloys as well as the pure metals. Consider for instance an alloy of iron and cobalt. Each of these elements would have about 0.7 electron per atom in the 4*s* state, so that we can assume that the alloy has the same. Then the average number of 3*d* electrons per atom will be somewhere between 7.3 (as in iron) and 8.3 (as in cobalt), the exact amount depending on composition. It is this number which we have used as the abscissa in the plot. The graph shows that the magnetic moment is almost precisely what we should get for a pure metal (if it existed) with the number of 3*d* electrons per atom which the alloy has. In other words, the only effect of going from a pure metal to an alloy, as far as this property is concerned, is to change the average number of electrons per atom. We see that this result holds, not merely for the iron-cobalt alloys, but for Fe-Ni, Ni-Co, Ni-Cu, Ni-Zn, Fe-V, Fe-Cr, all of which lie practically on a single line. This line corresponds to our rule that the number of empty states in the 3*d* shell gives the magnetic saturation, up to a point between iron and cobalt; above this value, the saturation moment falls off, as we should expect.

The method which we have used here, of plotting all the alloys as a function of the number of empty spaces in the 3*d* shell, might be a useful concept in other applications of the theory to alloys. Of course, many properties, particularly those depending on the entropy change in mixing unlike atoms, depend on other things in addition to the number of electrons per atom, but still it should be a useful method for correlating experimental facts. For instance, we can consider the phase diagrams of the binary alloys of Fe, Co, Ni and neighboring elements from this

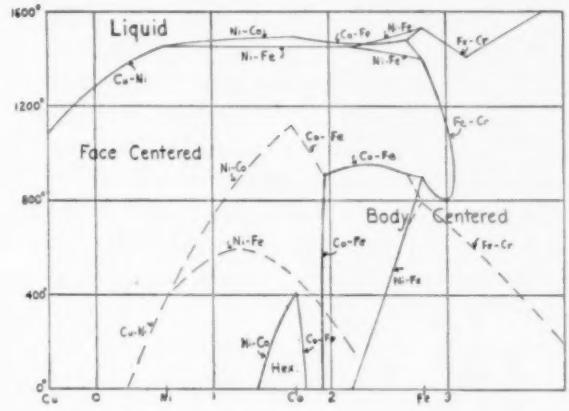

FIG. 4. Phase equilibrium diagram, as function of number of vacant spaces in shell of 3*d* electrons, for binary alloys Cu-Ni, Ni-Co, Ni-Fe, Co-Fe, Fe-Cr.

point of view. Fig. 4 shows the more important equilibrium lines for Cu-Ni, Ni-Co, Ni-Fe, Co-Fe and Fe-Cr, plotted against this same parameter, the number of empty spaces in the 3*d* shell. Certain general features become clearer in this way than by considering the separate phase diagrams. Thus the liquidus is a fairly smooth curve, rising toward the middle of the transition group at Cr (according to our previous statement that the most tightly bound elements tend to come toward the middle of such a group). In a broad way, the alloys with less than about two empty spaces in the 3*d* shell are face centered, those with more are body centered. The magnetic transformation, or Curie point (shown by a dotted line), starts at $-273°C$ for zero empty spaces in the 3*d* shell, the point at which ferromagnetism starts, then rises to a maximum at something like two empty spaces, falling again in the region of the body centered lattice in much the same way that the magnetic saturation value falls. But the details are very complicated. In particular, the properties of cobalt are quite different from those of an iron-nickel alloy with the same number of electrons per atom. The transition lines are all much higher for cobalt than for the Fe-Ni alloy, and the hexagonal phase of cobalt appears, analogous perhaps to a hypothetical hexagonal phase of Fe-Ni which

might appear at temperatures too low to reach. The reason for this is not perfectly clear, but presumably is connected with the entropy difference between the pure cobalt, where there is no entropy of mixing, and the Fe-Ni alloys, which has such an entropy effect. This method of considering the whole series of alloys as far as possible as a one-parameter family has not been used much up to the present, but may be valuable in visualizing regularities in the series which might otherwise escape attention.

Of course, the methods just described can be used only for substitution solid solutions, in which the substituted atoms are almost of the same size. This is more nearly true in the iron group, perhaps, than anywhere else in the periodic table. But we must not lose sight of the fact that a great deal of the variety in the structure of alloys arises on account of the combination of atoms of quite different sizes. The theory of energy bands suggests, as we should expect, that structures consisting of two or more types of atoms of different sizes would tend to form compounds rather than purely metallic solid solutions. The reason is essentially that valence electrons in one type of atom do not have ordinarily the right energy to act at the same time as valence electrons in the other type. In a pure metal, or an alloy of very similar atoms, of course this difficulty does not appear. But if the electrons in one atom are not free to wander at will to their neighbors, the average kinetic energy is decreased, there is less freedom of movement and poorer conductivity, and the energy bands tend to narrow down, approaching the situation which we have seen is to be identified with the compound. At the same time the interatomic forces become more like valence forces, more dependent on the exact arrangement and spacing of the atoms, and the structure tends to become hard and relatively brittle, rather than soft and ductile like the more typical metals. Such behavior, of course, is typical of metals of the iron group alloyed with quite different elements, as carbon, silicon, and nitrogen. The theory of such alloys has not been touched as yet but we may look forward to important results when it can be worked on.

From what we have seen, the theory of metals and alloys is still in an early stage, where it can suggest rather than work out the answers to problems. Its promise, nevertheless, seems to be great. Thermodynamics has been the great guide of metallurgy in the last generation. But thermodynamics has great weaknesses. As far as it goes it is correct, but it can give us very little insight into the mechanism underlying the properties of a physical or chemical system. To get further, we must have a model of the system, whose properties can be worked out from first principles, and from which, by use of the methods of statistical mechanics, we can derive values of the quantities which thermodynamics must postulate as empirical facts. For the first time, the theory of metals is able to see its way clear to provide such models, accurate enough so that we can really hope to get quantitative information from them without a prohibitive amount of work. It does not seem fanciful to hope that this theory will bring about as great advances in theoretical metallurgy in the next generation as thermodynamics has provided in the last one.

Research is the reconnaissance party of industry, roving the unknown territories ahead independently, yet not without purpose, seeing for the first time things that all the following world will see a few years hence.—S. M. KINTNER

What Electrons Can Tell Us About Metals*

By C. J. DAVISSON

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LET me begin by quoting one or two sentences from Professor Van Wert's new and admirable *Introduction to Physical Metallurgy*. Professor Van Wert says that "Progress in this search" [the search "to discover the essential nature and meaning of metal behavior"] "has been beset by many obstacles. Experimental inquiry has often been hampered by the lack of adequate research tools, for in metal research as elsewhere investigative enterprises have, on occasion, had to wait on the invention or development of more precise analytic or measuring appliances or methods."

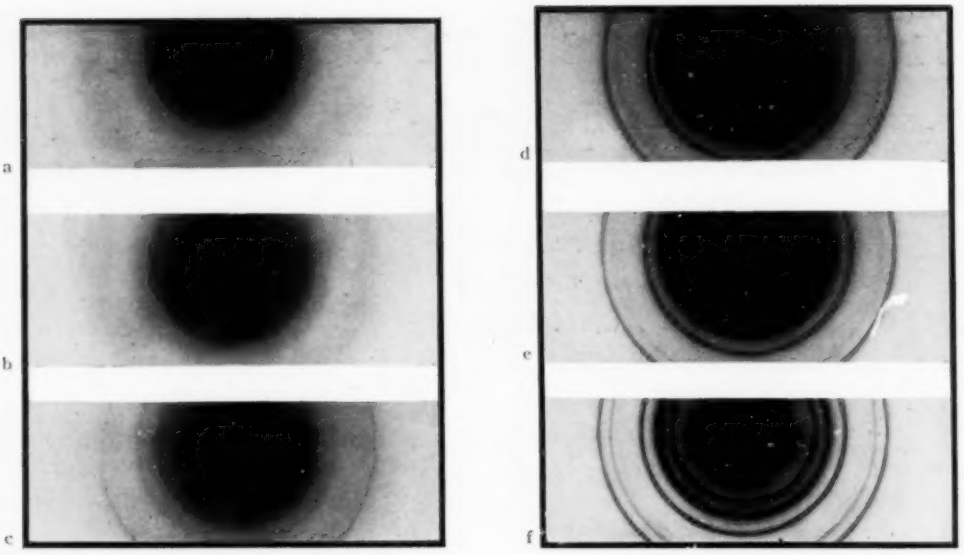
I quote this statement of Professor Van Wert's because it is of a new research tool that I am to speak, a tool the lack of which has hampered certain types of experimental inquiry in metallurgy up to the present time. I speak of electron crystal analysis as a new research tool because, although 10 years of age, it seems not to be well known to metallurgists. It is true that metallurgists here and there have made use of it. But it is true also that no mention is to be found of either electron diffraction or of electron crystal analysis in any book on metallurgy; at least there is no mention of either of these subjects in any one of four books on metallurgy reviewed in *Nature* only last month. It is fair to say, I think, that electron crystal analysis is not as yet a recognized tool of metallurgical research.

This simplifies matters for me, for it allows me to treat the subject in as elementary a fashion as I please. What I have thought of doing is first to make some general statements about electron waves and electron diffraction, then to describe a few typical investigations in which electron diffraction has been employed, and finally to say something about the technique of this new type of crystal analysis.

The most general statement one can make in regard to electron diffraction is perhaps that it is very similar to x-ray diffraction. A beam of electrons in which the electrons all travel with the same speed has a definite wave-length just as has a beam of characteristic x-rays. The wave-length of a beam of electrons is inversely proportional to the momentum of the individual electrons which compose it. The present practice in electron crystal analysis is to use electrons which have been accelerated through from 30 to 70 kilovolts. For these the wave-length range is from 0.07 to 0.05 angstrom units. The wave-length of the $K\alpha_1$ radiation of tungsten is greater than 0.2 angstroms and this is the shortest wave regularly used in x-ray crystal analysis, so that one difference between electron crystal analysis and x-ray crystal analysis is that in the former we work with shorter waves.

But the really great difference between x-rays and electrons is in the extent to which they are scattered and absorbed by matter. The characteristic for which x-rays are most celebrated is their power to penetrate thick slabs of material entirely opaque to ordinary light. Associated with this transparency which all materials exhibit to a greater or less extent, is a correspondingly low scattering power. The specimens examined by x-ray diffraction must not be too thin or the radiation scattered into the pattern will be negligible compared with that remaining in the primary beam.

* Presented at the Symposium on Metals, Cambridge, Massachusetts, January 28–30, 1927.



a, as prepared.
b, heated 10 hours at 690°C.
c, plus 10 hours at 680°C.

d, plus 50 hours at 680°C.
e, plus 2 hours at 800°C.
f, plus 1 hour at 1000°C.

FIG. 1. Electron diffraction patterns of oxide film from aluminum—by Preston and Bircumshaw.

With electrons the case is entirely different. All substances are more opaque to electrons of the wave-lengths used for crystal analysis than are metals to visible light. Electrons are no more suitable for examining sheets of metal by transmission than metal sheets are suitable for replacing glass in windows. To be suitable for examination by electrons by transmission, a specimen must be no more than a few hundred angstroms in thickness It must be just the sort of specimen which cannot be examined by x-rays.

Massive specimens can be examined by electrons by reflection. The beam is directed onto the surface at near grazing incidence, and the half-pattern which is produced reveals the crystalline state of a surface layer of excessive thinness or, more often, the crystalline state of tiny points and edges which protrude from the surface and transmit parts of the incident beam. Invisible films of material, different chemically from the bulk of the specimen, are frequently discovered by this method.

The reflection method has, of course, long been used in examining metals by x-rays. Here too the patterns are determined by material lying near the surface. But the layer contributing to the pattern is never so slight for x-rays as it is for electrons. An invisible surface film would never be found by x-rays; the pattern it produced would be swamped by the much stronger pattern due to the underlying metal. Thus x-rays and electrons are complementary in their uses, rather than coincident. Gossamer films is the province of electrons—relatively thick and bulky specimens is the province of x-rays.

I will now describe very briefly three out of some 700 investigations which have been made by electron diffraction during the last ten years. Of the three I have chosen, one only has to do with ferrous metals, but all will illustrate the kind of information a ferrous metallurgist might expect to obtain concerning his materials.

The first is an investigation made by Preston and Bircumshaw[1] in the National Physical Laboratory. Preston and Bircumshaw have been using electron diffraction for some time to study the oxidation of metals. In their most recent paper they describe their investigation of the oxide film which forms on aluminum at room temperature. They tried first the reflection method using as specimens small plates cut from cold-rolled sheets. The patterns they obtained

[1] Phil. Mag. **22**, 654–665 (1936).

consisted of portions of two broad fuzzy rings such as are produced when x-rays are diffracted by a liquid. This might mean that the oxide formed on aluminum at room temperature is amorphous, or it might mean that the surface metal had been rendered amorphous by the rolling and that the pattern was due to the metal rather than to the oxide. To dispose of this uncertainty Preston and Bircumshaw dissolved away thin sheets of aluminum by gaseous HCl, leaving only the oxide films. These they examined by transmission and again they found the broad fuzzy rings. The oxide film formed on cold-rolled aluminum at room temperature is undoubtedly amorphous.

On heating, the film remains amorphous to about 700°C at which temperature a pattern of sharp rings begins forming. When fully formed the pattern is that of $\gamma-Al_2O_3$. It is identical with the pattern produced by oxide films fished from molten aluminum. Patterns obtained at various stages of the transition from amorphous to crystalline oxide are reproduced in Fig. 1. There are certain peculiarities about the early stages of the crystallization, as revealed by the patterns, which are not yet understood.

face after abrasion—more specifically, after the face had been smoothed by filing. The results which Dr. Germer obtained are perhaps of some interest to metallurgists because of a similarity between what happens when a galena crystal is filed, and what happens when a metal is hardened by cold working.

To understand the patterns produced by the abraded crystal one must understand first the pattern produced by the normal or undisturbed crystal. Galena forms crystals of the rocksalt type—a checkerboard of lead and sulphur atoms extended into three dimensions. The crystal cleaves on its cube face. The reflection pattern produced by this face when well etched is under most circumstances an array of spots. If the primary beam, always at grazing incidence, lies near [100] direction in the surface, the array is square, Fig. 2. A score or more of diffraction beams issue from the crystal with the crystal and incident beam in fixed relation to one another, which may seem rather surprising to those familiar only with x-ray diffraction.

There are two circumstances which contribute to this production of multiple beams; the first is that the wave-length of the electrons is quite

FIG. 2. Cleavage face thoroughly etched—beam parallel to [100] direction.

FIG. 3. After abrasion (filing at right angles to primary beam).

FIG. 4. Same after etching (pattern due to undisturbed crystal showing through that due to rolled crystallites).

FIGS. 2, 3, and 4. Electron diffraction patterns produced by a galena crystal—L. H. Germer.

Next I will describe an investigation made by Dr. L. H. Germer[2] of the Bell Telephone Laboratories. The material investigated was galena in the form of single crystals. One of the matters studied was the crystalline state of the cleavage

small compared to the constant of the crystal, and the second is that there is an appreciable misalignment among the crystallites which form the galena crystal. The electron wave-length is less than 1/100 of the linear dimension of the crystal cell. The Bragg angles are correspondingly small. The smallest of them are considerably

[2] Phys. Rev. **50**, 659–671 (1936).

smaller than the largest angles by which the crystallites fail of perfect alignment. In these circumstances Bragg reflections of differing indices can occur from different crystallites for one and the same setting of beam and crystal. This is an explanation, though not the only possible explanation, of the spot pattern observed in this case. For present purposes we need know only that this square array of spots is the pattern produced by the cleavage face of the undisturbed crystal when the incident beam is parallel to a cube edge direction, or nearly so.

The crystal face was then abraded—filed parallel to a [100] direction. In this condition it produced a pattern of rings, Fig. 3. The surface material had been broken up and skewed

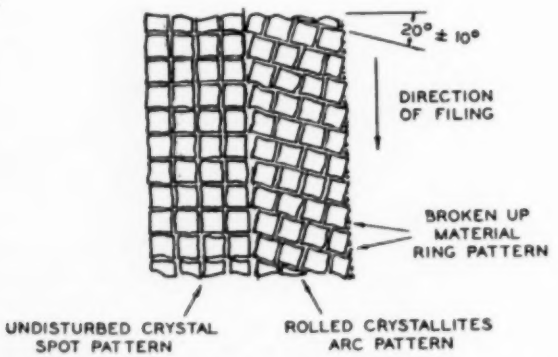


FIG. 5. Schematic cross section of a galena crystal illustrating the deformation produced by abrasion.

about into random orientations. What was observed was the powder pattern of galena.

The surface of the crystal was then etched away a little at a time, and the pattern recorded at each stage of the etching. What was observed depended upon the relation between the direction of the primary beam and the direction of filing. When the direction of the primary beam was at right angles to the direction of filing the ring pattern weakened, some of the rings vanished completely and others were reduced to short arcs. This pattern of arcs, which exhibited no recognizable symmetry, persisted without alteration through some stages of etching. Then the square array of spots began showing through, indicating that undisturbed crystal was beginning to outcrop, Fig. 4. When this occurred it was discovered that each of the arcs was associated with one of the spots, and that the arc pattern could be described as a rotation and streaking out of the spot pattern. It was as if the crystallites in a layer immediately below the surface had been rolled by the file through angles ranging from 10 to 30 degrees, and this undoubtedly is what had happened.

The behavior is illustrated in Fig. 5. The drawing represents a cross section of the crystal after filing, the direction of filing being downward. The surface layer is badly broken up and disorganized and produces the ring pattern. Below this is a relatively thick layer in which the crystallites, represented as blocks, have been rolled by the file. They roll from 10 to 30 degrees and then jam. The material in this layer produces the arc pattern. Below this is the undisturbed material which produces the spot pattern. This exemplifies rather well one of the actions which is known from x-ray analysis to occur in work hardening.

The last experiment I will describe is by Dr. H. R. Nelson[3] of the Battelle Institute. Dr. Nelson has vaporized iron onto flat surfaces in vacuum and has studied the patterns produced by these deposits before and after exposure to air. One of these is reproduced in Fig. 6. Most of the spots or short arcs of this pattern fall on equally spaced lines parallel to the surface of the specimen. These alone are due to the iron and what they reveal is that the iron forms crystals as it is deposited, and that a [111] direction of each of the crystals stands normal to the general surface of the specimen, or nearly so. The orientation is that preferred by iron when deposited electrolytically.

The spots and short arcs which do not fall on the equally spaced lines are due to oxide formed on the iron crystals during an accidental 5 minute exposure of the deposit to air at room temperature and at a pressure of about 10^{-3} mm of mercury. The fact that the oxide forms a symmetrical pattern of spots rather than a pattern of rings means that like the iron crystals the oxide crystals have a strongly preferred orientation.

With further exposure to dry air, Fig. 7, the oxide pattern became stronger, the iron pattern weaker and eventually, Fig. 8, only the oxide

[3] J. Chem. Phys. **5**, 252 (1937).

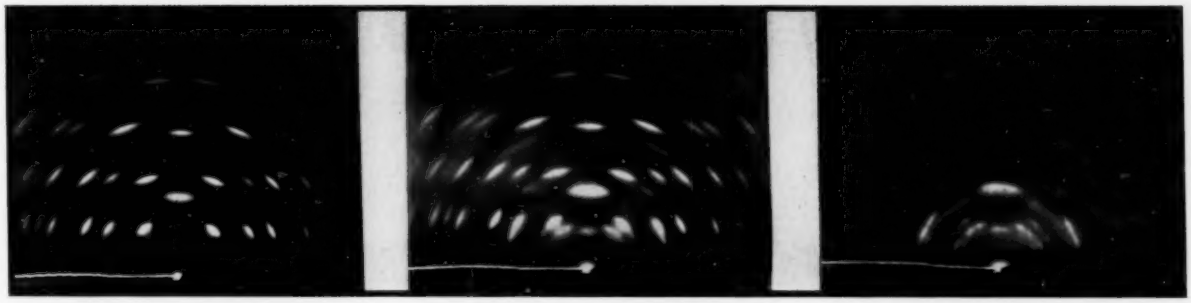


FIG. 6. Oriented film slightly exposed to air.

FIG. 7. After 2 minutes exposure at room temperature.

FIG. 8. After 15 minutes exposure at 200°C.

FIGS. 6, 7, and 8. Electron diffraction patterns produced by a vaporized deposit of iron during oxidation—H. R. Nelson.

pattern remained. Dr. Nelson has identified the structure of the oxide as that of magnetite and the preferred orientations as that in which a $[\sqrt{2}, 1, 0]$ direction in the crystal stands normal to the surface.

It seems odd, unlikely even, that there should be anything unique about the $[\sqrt{2}, 1, 0]$ direction, for it is not, so to speak, a crystallographic direction—these have in all cases rational indices. The explanation is, however, quite simple. Mehl, McCandless and Rhines[4] have shown that when iron is oxidized at high temperature, the orientation of the oxide crystals is related in a simple way to the orientation of the iron crystals upon which they are formed. We stand the oxide crystal cube squarely upon the iron cube, rotate the oxide cube 45 degrees about a vertical axis, and we have the relation between the two structures. One feature of this relationship is that every [111] direction in the iron structure is parallel to a $[\sqrt{2}, 1, 0]$ direction in the oxide structure. It follows that if the oxide crystals formed at room temperature build onto the oriented iron crystals in the same way as those formed at high temperature, the oxide crystals will indeed stand with $[\sqrt{2}, 1, 0]$ directions pointing directly outward from the surface.

These experiments illustrate some of the things electrons can tell us about metals. They can give us information about the chemical and crystalline constitutions of metallic surfaces and of thin films of metals and their compounds. These, I believe, are not matters to which metallurgists are entirely indifferent.

Let me say a few words now about the means employed in obtaining electron diffraction patterns such as we have seen. The material requirements are a source of electrons, a supply of constant high potential, a set of apertured electrodes for use in accelerating the electrons and forming them into a narrow beam, an adjustable mounting for holding the specimen, a fluorescent screen for use in adjusting beam and specimen, and a photographic plate for recording the pattern. These parts except the voltage supply are sealed into a gas-tight chamber, which is exhausted and held by continuous pumping at a pressure of 10^{-4} mm of mercury or less.

These features are found in all electron diffraction equipments, but the form they take varies considerably from one equipment to another. Almost every experimenter works with a camera of his own design which he regards as superior in some respects to all others.

A diagram of the diffraction camera, used by Finch, Quarrell and Wilman[5] at the Imperial College of Science and Technology, is shown in Fig. 9. The source of the electrons is a low pressure gas discharge which is maintained in the chamber at the top. Cathode rays from this discharge enter the main chamber through a narrow tube and form the electron beam. Halfway down the tube is the specimen holder, and at the bottom the fluorescent screen and photograph plate, which are manipulated by mechanical controls passing through the base. An important feature of this equipment is a coil

[4] Nature **134**, 1009 (1934).

[5] Trans. Faraday Soc. **31**, 1051–1080 (1935).

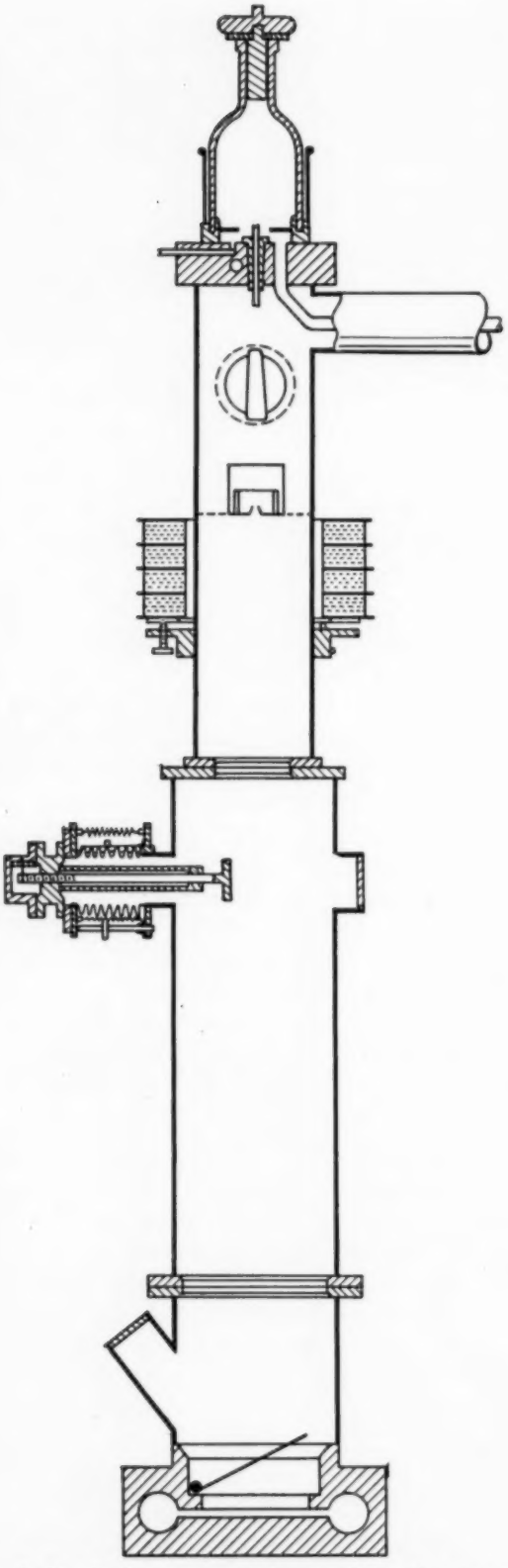


FIG. 9. Gas discharge type of electron diffraction camera—Finch and Quarrell.

magnet which surrounds the tube between source and specimen and is used to focus the beam on the photographic plate. It is used also, by tilting it, to align the beam. The position and orientation of the specimen are adjustable from outside through gas-tight bearings and a metallic bellows. The parts shown near the top of the main tube are used for deflecting the beam on and off the photographic plate, and for measuring the beam current. The beam current used in this and other cameras is usually a small fraction of a microampere. The exposure times range from a tenth of a second or less to several seconds, depending on the beam power and the nature of the specimen.

The main chamber is pumped continuously and necessarily rapidly since gas enters it continuously from the discharge chamber. This camera or one of the same type is now manufactured for sale by the Cambridge Instrument Company. It is capable of giving very fine results.

Another embodiment of the same essential features is illustrated in Fig. 10. This is a diagram of a camera developed at the Bell Telephone Laboratories. The electrons come from a tungsten filament at A, and are accelerated from A to B. The beam is defined by small apertures in parts B and B'. The alignment of the beam is achieved by adjustment of the positions of the apertures through sylphon bellows and by the use of crossed electrostatic fields in the region between B and B'. The specimen, in a mounting adjustable through bellows, is placed just beyond the second aperture. The fluorescent screen closes the end of the main tube. The photographic plate is held until wanted in the upper part of the vertical cylinder shown in the drawing. It can be lowered into position for exposure or withdrawn by gas-tight mechanical couplings to the outside. Adjustable stops are set to limit the amount of plate exposed. With exposures limited to strips one or two centimeters wide, portions of 10 or 15 patterns can be recorded on a single plate. This camera is in daily use on corrosion and other problems arising in the telephone plant.

I hardly need point out that what with having to open and reseal the camera to change specimens and plates, and what with having to worry

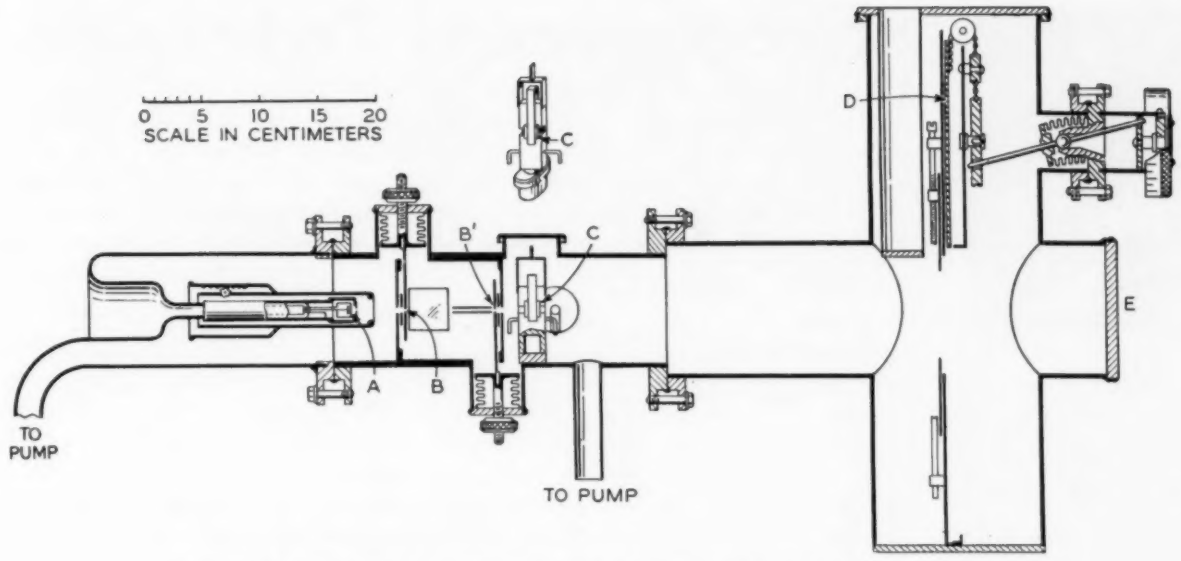


FIG. 10. Filament type of electron diffraction camera—Bell Telephone Laboratories.

about pressures and pumps, the recording of electron diffraction patterns is more bothersome than the recording of x-ray patterns. And yet when the equipment is properly installed and in continuous service the technique is much less difficult than might be imagined. Dr. Nelson reports that with his apparatus he goes from vacuum to vacuum in from 4 to 7 minutes.

It is rather pointless, however, to make comparison of the two techniques, for there are few investigations in which one may use x-rays or electrons as he chooses. Either the investigation is one to which x-rays alone are applicable, or it is one to which only electrons are applicable. If the latter, one has no choice but to accept the experimental difficulties, such as they are, in the usual dauntless spirit.

Metallurgists have found x-ray diffraction applicable to so many of their problems it will be surprising, indeed, if, in the near future, they do not find electron diffraction applicable to many others.

The application and development of scientific ideas in industry depend upon a complete understanding of the way in which industry can make use of science and scientific method, and this can only be achieved if the problem of cooperation is studied by the man of science as well as by the industrialist. The differences of experience, training and outlook between men of scientific education and many others engaged in productive industry are not always considered by those who are anxious to secure the potential benefits which science, wisely used, can bring to the community. The scientific worker has to meet the industrialist half way.

Science, Industry and Society
(Editorial in *Nature*, March 27, 1937)

Electrochemical Techniques *in* Corrosion Study*

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New York, New York

THE chemical reactions which occur in the corrosion of metals in the ordinary atmosphere are often complex and varied. They depend upon the composition, physical state and surface condition of the metallic material, as well as upon the chemical components of the environment,—their phases and concentrations. These reactions are affected by temperature and temperature fluctuation, by the movement or circulation of the air (or electrolyte in case of submerged corrosion), by the nature and the solubility of the corrosion products and particularly by the position in which solid corrosion products are precipitated with reference to the surface of the corroding metal. Moreover, in actual cases of corrosion, marked changes may occur in the nature and concentrations of the surrounding nonmetallic reactants during the process of corrosion. It is apparent that a rather detailed knowledge of both the metal and of the environment in which it is to be used are essential in the chemical investigation of corrosion problems.

The engineering need of information of the relative durability of metallic materials has led to the widespread use of corrosion tests. These vary widely in character from exposure for prolonged periods to the actual environments in which the metals are to be used to laboratory tests of great variety. Such tests, while of considerable assistance in the selection of suitable materials for given purposes, have not generally furnished sufficient information as to the processes of corrosion to permit prediction as to what will happen under slightly different conditions—the problem which usually presents itself.

The relatively long period of time required to obtain measurable results on metal specimens exposed to natural environments led to an interest in accelerated corrosion tests in which the rates of reaction are accelerated by increasing the temperature, the humidity, or by otherwise modifying the natural environment. It soon became apparent, however, that the results of accelerated tests could not be correlated usually with actual rates of corrosion under the conditions in which the metals are used, and moreover, that the quality ratings established by such tests were often erroneous. Except for certain limited uses, the idea of accelerated corrosion testing has been largely abandoned.

The more recent trend in corrosion investigations has been in the direction of the study of corrosion processes with experimental techniques of sufficient sensitivity to measure the actual rates of reaction occurring in natural environments. In this way significant information can be obtained in a reasonable time without the danger of distortion introduced by accelerating the corrosion process. The present paper describes the application of some of these methods and what they reveal concerning the mechanisms of corrosion processes.

In recent years considerable progress has been made in the elucidation of corrosion reactions. It has been well established, for example, that corrosion in the presence of moisture is an

* Presented at the Symposium on Metals, Cambridge, Massachusetts, January 28–30, 1937.

electrolytic process in which the metal dissolves at certain areas or points—the anodes of small corrosion cells, the cathodes of which are the adjacent areas on the metal surface at which hydrogen is deposited. The driving force of these cells arises either from some chemical or physical inhomogeneity of the metal or some inhomogeneity of the environment. Their electrolytic operation is influenced by the composition, size and distribution of the anodic and cathodic areas, by the character of the corrosion products and by the chemical nature and the conductance of the surrounding environment. This statement of the replacement of hydrogen from the contiguous electrolyte by the dissolution of the corroding metal does not present a detailed view of the actual mechanism which must involve the action of the solvent as well as the exchange of electrons at electrode areas. The essential fact is that metal ions replace hydrogen ions in the electrolytic environment. A second step of equal importance in the process lies in what happens to the electrodeposited hydrogen since the continuation of the attack depends upon its disposal. It may be evolved as molecular hydrogen by metallic cathodes of low overvoltage, or it may be removed by combination with oxygen or other oxidants. In either case, when the rate of corrosion depends upon the rate of hydrogen disposal, the process is said to be cathodically controlled. Similarly, when the anodic reaction controls the rate of corrosion, as, for example, in the corrosion of magnesium in water where the rate of corrosion depends upon the rate of discharge of hydroxyl ions,[1] the process is said to be anodically controlled. In other words, the progress of corrosion may be controlled by the extent of polarization at either cathode or anode or at both. Either process may reduce the effective potentials of the corrosion cells virtually to zero.

It is of interest to consider the manner in which various environmental constituents influence the current voltage or polarization characteristics of corrosion cells. Reference has already been made to the depolarizing influence of oxygen at cathode areas. In this function it is the most important promoter of corrosion. While some hydrogen is usually evolved in the corrosion of most metals (owing often to the presence of impurities of low overvoltage) the extent of attack depends mainly in most cases upon oxygen availability. Nitrates and other oxidants play a role similar to that of oxygen. The rate of cathode depolarization and therefore the rate of corrosion in these cases depends upon the rate at which oxygen or oxidants reach the cathode areas. This is usually a function of the concentration of these substances and of temperature and is largely influenced by convection currents. There are certain substances, however, which if present in the environment may hinder cathodic depolarization. Gelatine, diethylamine, hydroquinone, and various colloidal or reducing substances inhibit the corrosion of iron in acids because of their interference with the cathodic reaction. Colloidal soils, experience shows, may retard corrosive action by assisting in the maintenance of hydrogen films at cathodic areas,[2] or by otherwise reducing the accessibility of oxygen to these regions.

Anodic processes in corrosion cells consist in either the dissolution of metal ions or the precipitation of anions at the metal surface. An increase in metal ion concentration or the accumulation of solid corrosion products is reflected in anodic polarization. The solubility of compounds of the metal and anions of the solution largely determines the course of the reaction. Compounds of low solubility tend to form films over anodic areas. The hydroxides (or hydrated oxides) of most metals fall in this class as do salts such as the silicate or sulfate of lead, and phosphate of iron. When, however, the corrosion products are more soluble, other constituents of the environment may, by means of secondary reactions, precipitate a solid phase at an appreciable distance from the metal surface. Thus, in the corrosion of iron, the moderate solubility of ferrous hydroxide permits migration of ferrous ions from the surface to an adjacent zone where, owing to the action of dissolved oxygen, the less soluble ferric hydroxide may be precipitated as a rust scale. Similarly, lead carbonate may be precipitated in the body of the solution when lead corrodes in distilled water exposed to the air. In neither of these cases does the precipitation of the solid phase interfere

[1] Whitby, Trans. Faraday Soc. **29**, 1318 (1933).

[2] Burns and Salley, Ind. Eng. Chem. **22**, 293 (1930).

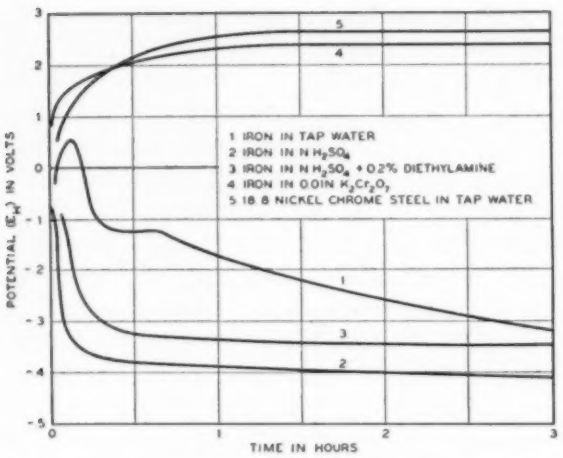
1 IRON IN TAP WATER
2 IRON IN N H_2SO_4
3 IRON IN N H_2SO_4 + 0.2% DIETHYLAMINE
4 IRON IN 0.01N $K_2Cr_2O_7$
5 18-8 NICKEL CHROME STEEL IN TAP WATER
POTENTIAL (E_H) IN VOLTS
TIME IN HOURS

FIG. 1. Potentials of iron and stainless steel in water and certain solutions.

with the anodic reaction—i.e., induce anodic polarization—or interfere with corrosion.

As indicated above, anodic polarization is most readily observed in solutions containing strong oxidants or anions which form relatively insoluble salts with the metal. It is conceivable that products comprising these surface films have a lesser solubility than the analogous compounds in bulk.[3] The process of anodic polarization of a metallic surface is a progressive action proceeding from the more anodic to the less anodic regions until the surface becomes substantially film-coated or passivated.

From the foregoing discussion it will be apparent that a knowledge of the electrochemical polarization characteristics of corrosion cells would indicate the nature of the process and make possible the prediction of corrosion behavior. It is usually impossible, however, to measure the current density potential relationships of individual corrosion cells owing to a lack of knowledge of the electrode areas involved. Actually, these areas are probably of a wide range of sizes and change in size during the progress of corrosion. Even the nature of the cathodes is often unknown. Practically, however, it is a simple matter to determine the composite of the resulting potentials and their change with time. If the observed potential of the metal surface as a whole tends to become more electronegative with time, it is indicative of increased cathodic polarization and the process is probably cathodically controlled. If, on the other hand, the trend of potential with time is in the electropositive or more noble direction, anodic polarization and anodic control of the process is apparent. These time-potential measurements may be readily made with a recording potentiometer.

In Fig. 1 there is shown the actual time-potential behavior of specimens of iron and 18–8 nickel-chromium steel immersed in water and certain solutions. It will be observed that the potential of iron in normal sulfuric acid rapidly becomes electronegative. The initial trend of a similar specimen immersed in distilled water is toward the noble side, owing probably to the passivity of the air-formed film. Within a few minutes, this trend is reversed, the specimen becoming more electronegative probably because of the destruction of this film. In the presence of an oxidizing solution, potassium dichromate, it will be seen that the potential of iron (represented by curve 4) becomes markedly electropositive. A similar passive behavior is shown by stainless steel in tap water (curve 5). The iron specimens, the potentials of which are represented by curves 1 and 3, displayed active corrosion, while iron in potassium dichromate solution (curve 4) and stainless steel in water did not corrode. In other words, a trend in the electronegative direction, which, as before mentioned, is characteristic of cathodic polarization, is associated with active corrosion, while the electropositive trend of potential is indicative of passivation. It is of interest to note in this connection that certain organic compounds used as corrosion inhibitors which, as before mentioned, function by retardation of the cathodic reaction appear to have little or no influence on the characteristic of time-potential curves. The experiment represented by curve 3, for example, differs from that of curve 2 only by the presence of 0.2 percent diethylamine, a typical corrosion inhibitor. Corrosion has been greatly retarded in this case but the iron is still in the active or corroding class.

Another instance of the usefulness of time-potential studies in revealing information as to corrodibility is illustrated in Fig. 2.[4] An alloy of copper containing 25 percent iron had in previous

[3] Whitby, Trans. Faraday Soc. **29**, 415 (1933).

[4] Schumacher and Souden, Met. and Alloys **7**, 95 (1936).

tests in another locality shown satisfactory resistance to atmospheric corrosion, but upon exposure to New York City air it was severely corroded. The time-potential curves show a tendency toward passivity which is accentuated by providing greater accessibility to oxygen (compare the experiment in which specimen was immersed in a body of solution, represented by curve 3, with that in which the specimen was merely in contact with a film of solution, curve 2). Passivity is readily destroyed, as shown by the abrupt break in curve 2 upon the addition of a drop of dilute sodium chloride. From this evidence it seems likely that the less favorable behavior of the copper-iron alloy in New York may be due to the small chloride component of the atmosphere.

A rather unique method of determining the relative rates at which similar alloys tend to corrode is to compare the rates of passivation in the presence of film-forming constituents.[5] For example, when specimens of lead and certain lead alloys are immersed in solutions of 7 normal sulfuric acid exposed to the air the initial potential of lead breaks abruptly after varying periods of time to the potential of the cathodic areas. A comparison of the rates of sulfation of several grades of lead and certain lead alloys,

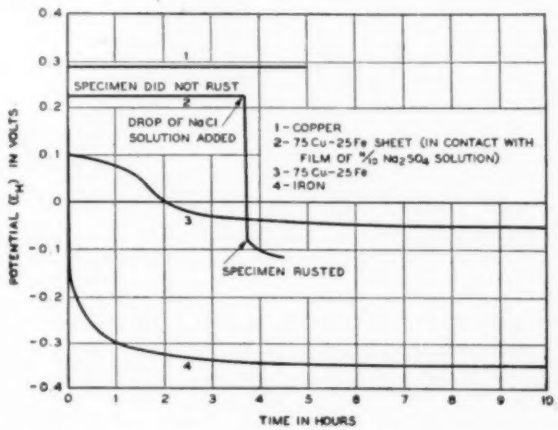
SPECIMEN DID NOT RUST
DROP OF NaCl SOLUTION ADDED
1 - COPPER
2 - 75 Cu - 25 Fe SHEET (IN CONTACT WITH FILM OF 1/10 Na2SO4 SOLUTION)
3 - 75 Cu - 25 Fe
4 - IRON
SPECIMEN RUSTED
POTENTIAL (EH) IN VOLTS
TIME IN HOURS

FIG. 2. Potentials of iron copper and an iron-copper alloy in dilute sodium sulfate solution.

determined in this manner, is given in Fig. 3. It should be emphasized that the relative surface activities of metals determined in this manner cannot be correlated directly to rates of corrosion in most actual environments because of the interference of film-forming substances which are usually present and which tend to block the primary reaction.

In the experiments described above it was found that a potentiometer of ordinary sensi-

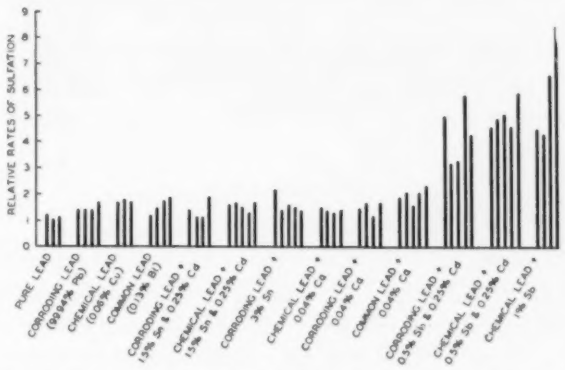
RELATIVE RATES OF SULFATION
PURE LEAD
CORRODING LEAD (99.94% Pb)
CHEMICAL LEAD (0.08% Cu)
COMMON LEAD (0.13% Bi)
CORRODING LEAD + 1.5% Sn & 0.25% Cd
CHEMICAL LEAD + 1.5% Sn & 0.25% Cd
CORRODING LEAD + 3% Sn
CHEMICAL LEAD + 0.04% Ca
CORRODING LEAD + 0.04% Cd
COMMON LEAD + 0.04% Ca
CORRODING LEAD + 0.5% Sn & 0.25% Cd
CHEMICAL LEAD + 0.5% Sb & 0.25% Cd
CHEMICAL LEAD + 1% Sb

FIG. 3. Relative rates of sulfation of lead and certain lead alloys in 7 normal sulfuric acid.

tivity was entirely adequate and that the recording feature was of considerable advantage in obtaining a complete record of potential with time. When, however, it became desirable to determine the corrosion behavior of an iron surface beneath a paint coating, a vacuum tube electrometer circuit proved essential for the measurement of the potential of the metal and its trend with time. By this means, it has been possible to determine whether the inhibitive action of primer pigments is due to chemical action or to their influence on the physical properties of paint films.[6] This work is being extended to include a wide variety of pigments.

Still another instrument which promises to be of considerable assistance in the study of passive film stability is the oscillograph. Fig. 4 (records 1 to 4, respectively) shows the behavior of the potential of passive iron as additional quantities (beyond 5 percent) of concentrated hydrochloric acid are added to the nitric acid solution responsible for the induction of passivity. It will be observed that the magnitude of the potential break increased with the concentration of the film-destroying chloride, and that the time required for repair of the film increased with the extent of the film failure. The instrument em-

[5] Burns, Bell Sys. Tech. J. **15**, 20, 603 (1936).

[6] Burns and Haring, Trans. Electrochem. Soc. **69**, 169 (1936).

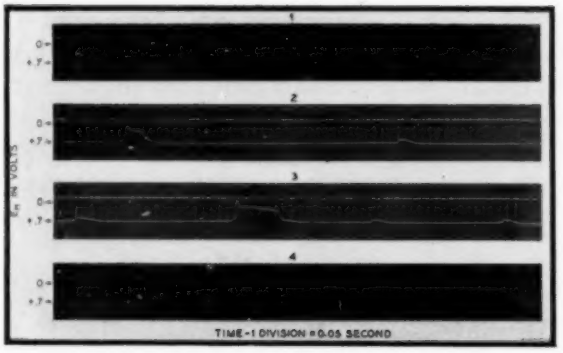

FIG. 4. Time-potential curve of iron in 16m HNO_3 upon adding 5–6 percent 12m HCl.

ployed in these measurements has a flat response characteristic up to 10,000 cycles, and would appear to be suitable for the study and comparison of protective films on various metals. Another investigator has recently reported results of an oscillographic study of the behavior of iron in chromate-sulfate solutions.[7]

These potentiometric studies show whether a metal is in a corroding or corrosion-resistant state, and this is the most important phase of the corrosion problem which presents itself in the commercial use of metals. As has been seen, such studies, while furnishing some indication of rates of corrosion, do not provide quantitative information on this point. The usual methods of determining corrosion rates—changes in weight or physical properties or measurement of the consumption of reactants—are usually laborious, and sometimes of low accuracy. Recently a new device of analytical chemistry, known as the "polarograph," has become available and preliminary experiments indicate that it should be useful in the determination of corrosion rates.[8] Essentially, it consists of a cell comprised of a capillary dropping mercury cathode, a mercury anode of relatively large area, and the electrolyte to be analyzed. In operation, an electromotive force is applied to the cell at a constant rate and the resulting current is automatically measured and recorded as a function of the polarizing voltage. With increasing voltage the current rises very slowly until the cathode potential attains a value at which a given positive ion is discharged (or reduced) at the mercury surface. At this point a rapid rise in current occurs, the magnitude of which is determined by the mobility and concentration of the ion. By providing a suitable (and constant) ionic environment (with lithium chloride in the experiments to be described) and employing a constant mercury dropping rate and capillary tip, it becomes possible to make rapid determinations of certain metals by comparison with solutions of known concentrations under the same conditions.

In Fig. 5 there is shown by means of polarographic measurements a comparison of the rates of corrosion in distilled water exposed to the air of one specimen each of chemical lead and

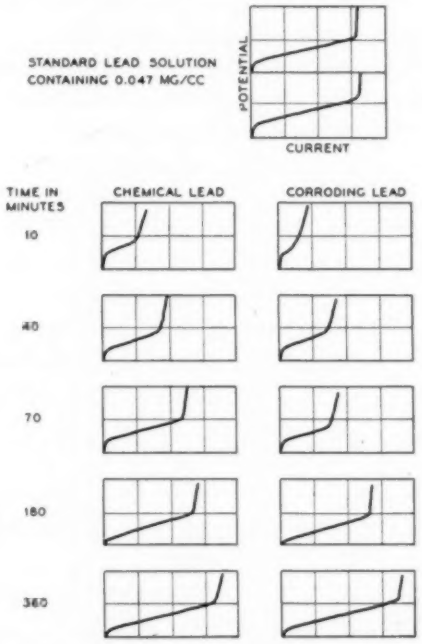

FIG. 5. Polarographic comparison of rates of solution of two grades of lead in distilled water at room temperature.

corroding lead. It will be seen that initially chemical lead corrodes more rapidly than corroding lead, which finding is in harmony with the results on their respective rates of sulfation contained in Fig. 3. It is of interest to note that after 360 minutes, the amount of corrosion indicated by the polarograms is nearly the same for both lead specimens, indicating that the reaction after its initial stages must be controlled by external factors rather than by the composition of the metal.

The polarographic method is capable of meas-

[7] Karschulin, Zeits. f. Elektrochem. **42**, 722 (1936).
[8] Heyrovsky, Mikrochem. **12**, 25 (1932); Kemula, Zeits. f. Elektrochem. **37**, 779 (1931).

uring one microgram of lead in a volume of one milliliter with a precision exceeding ±5 percent.

The high sensitivity of the method permits the measurement of the reaction rate in the initial stages of a corrosion process, thus contributing to the elucidation of the mechanism. In addition to speed and high sensitivity, another advantage of the method lies in the fact that two or more metal ions may be determined in the same measurement provided their deposition potentials (on mercury) differ by as much as 0.2 volt. It should prove to be particularly useful in investigations of the corrosion of alloys in which more than one component dissolves and where the nature of the process is obscure.

From the foregoing account it is seen that significant information concerning the nature of corrosion processes in the presence of moisture may be obtained by utilizing certain techniques of high sensitivity in place of the common and more cumbersome corrosion-life tests. Moreover, it is no longer necessary to resort to the practice of accelerating corrosion reactions in order to measure them. Observations of the potentials of materials in known environments and changes of these potential values with time which may be determined by means of a potentiometer or a vacuum tube electrometer show whether the metal surface is in a corroding or passive state. The existence and stability of passive surface films may be investigated advantageously by oscillographic measurements. Rates of corrosion may be measured readily by the use of the polarograph, the instrument being particularly suitable for determining the initial rates of reaction. These and other techniques of equal promise, such as electron beam diffraction and the newer methods of microchemistry, will be the means of important contributions to our knowledge of the corrosion of metals.

It seems to me a hopeless task to provide a complete and finished liberal education suitable to this century by four years of college work. The only worthwhile liberal education today is one which is a continuing process going on throughout life. It is from this premise, I suggest, that this question of "knowledge of wide surface and little depth" must be judged. I should reject all informational tests applied to recent graduates as indicative of the effectiveness of our general education. Whether a liberal education has been a success or failure should be measured by the student's breadth of vision fifteen or twenty years after graduation. Has the smattering acquired in college worn thinner with each succeeding year? If so, it has been of little value. Or has it provided a basis for continued intellectual and spiritual growth? In this case it has been the most significant part of the college training.

—JAMES B. CONANT

The Dynamics of A Bow and Arrow

By C. N. HICKMAN

New York, New York

A SERIES of articles by the author dealing with the static strains and stresses in a drawn bow have been published in *Ye Sylvan Archer*.[1] These articles have shown the effect of the shape and form of bending of the bow upon these strains and stresses. Some interesting and valuable information was obtained from that work which has materially changed the design of modern bows.

Considerable time has been spent by the author in studying, by experimental methods, the behavior of a bow while the arrow is being discharged. Some of the results of these investigations have been published in the magazine previously mentioned and in *The Journal of the Franklin Institute* (October 1929). He has also had high speed pictures (one to three thousand frames per second) taken by Electric Research Products which, in addition to putting an end to all controversies regarding the "archers parados," furnish an excellent means of studying the motions of the arrow and bow limbs.

While these experimental methods are interesting and valuable, they are necessarily slow and expensive. An analytical method was therefore developed for determining the dynamical forces, the velocities and accelerations of the arrow and bow limbs. This method has been useful in improving the design and cast of bows. The method is briefly set forth in this paper. Since the dynamical treatment depends on the static conditions, a few of the static equations will first be given.

Let us assume that the bow is made up of a short rigid middle section and two limbs, which, for all positions of the draw, bend in arcs of circles. This assumption is reasonable because practically all bowyers construct their bows so that they bend in this form. In the articles referred to above, it has been shown that a very simple design having this form of bending will stress all sections of the bow equally. It is believed that for most bows this form of bending is desirable. If for some particular bow another form of bending is desired, the method used here may be applied to it.

Referring to Fig. 1, let B equal one-half the length of the bow, L equal one-half the length of the rigid handle section, $B_1=B-L$ equal the length of the active bending portion of each limb, S equal one-half the length of the string, H equal the distance from middle of bow to middle of line connecting bow tips, P equal distance from arrow nock to middle of line connecting bow tips, $D=H+P$ equal length of draw, Y equal one-half length of line connecting bow tips, A equal angle between the line connecting bow tip and point O and the line representing the position of the undeflected limb. (The point O is located at a distance of $3B_1/4$ from the tip of the bow.) Let E equal angle made by string with the line connecting the bow tips, f equal the static force at each bow tip in a direction tangent to its path, T equal static tension in the string, F equal static drawing force, $Y_1=Y-B_1/4-L$ and N equal the distance along the path made by the bow tip during the draw.

It may be shown that the path made by the bow tip is part of a cardioid. The portion of the cardioid traversed by the bow tip is almost a perfect arc of a circle whose radius is $3B_1/4$ and whose center is located at a distance of $3B_1/4$ from the tip of the undeflected bow. Since the equations which are to be obtained are based on the assumption that the bow tips travel along

[1] Nov. 1930 to Aug. 1932 (published at Albany, Oregon).

arcs of circles having radii of $3B_1/4$, the accuracy of this assumption will now be investigated.

In Fig. 2, let B_1 represent the length of the bending portion of one of the bow limbs. Since the limb bends in the arc of a circle, B_1 is the arc of a circle having a radius of r and the angle θ corresponds to this arc. The chord $K=2r \sin \theta/2$ and $r=B_1/\theta$ where θ is measured in radians. Therefore $K=(2B_1/\theta) \sin \theta/2$. The point Q is selected so that $QY=3B_1/4$ and $QX=B_1/4$. $XY=B_1$ and represents the length and position of the active bending portion of the limb before the draw. The angle $QXZ=\theta/2$. Let $QZ=R$. Then:

$$R=[B_1^2/16+K^2-(KB_1/2) \cos \theta/2]^{\frac{1}{2}}.$$

Substituting the above value of K and making use of well-known trigonometric relations it may be shown that:

$$R=(3B_1/4)(1-16\theta^4/9\underline{|6}+\cdots)^{\frac{1}{2}}.$$

Since the angle θ for a bent bow never exceeds unity (i.e. 57.3°), all the terms under the radical except the first one are negligible. Therefore $R=3B_1/4$. (The maximum value of θ for the average bow is equal to about $\frac{2}{3}$. For this value of θ, the second term under the radical would equal 16/32,805. Neglecting this term results in an error of less than two parts in 1000.)

Referring again to Fig. 1, the force f is proportional to the displacement N and is therefore proportional to the angle A which corresponds to the arc N.

Let $f=CA$ where C is a constant depending on the dimensions and material of the bow.

A subscript 0 will be used to indicate the value of any of the above variables when the bow is in its braced but undrawn position.

The following trignometric and algebraic relationships may now be obtained:

$$Y_1=(3B_1/4) \cos A, \qquad Y=Y_1+B_1/4+L,$$
$$S=Y_0=(3B_1/4) \cos A_0+B_1/4+L$$
$$=\tfrac{1}{2} \text{ length of string},$$
$$H=(3B_1/4) \sin A, \qquad H_0=(3B_1/4) \sin A_0,$$
$$E=\cos^{-1} Y/S, \qquad P=S \sin E=(S^2-Y^2)^{\frac{1}{2}},$$
$$D=H+P, \quad T=f/\sin (A+E)=CA/\sin (A+E),$$
$$T_0=f/\sin A_0=CA_0/\sin A_0,$$
$$F=2T \sin E=(2AC \sin E)/\sin (A+E),$$
$$N=3B_1A/4, \qquad N_0=3B_1A_0/4.$$

Dynamical Treatment

Let x represent the distance traveled by the arrow in t seconds. Then: $dx/dt=V$ where V is the instantaneous velocity of the arrow. But $x=D'-D$ where D' is the value of D for the fully drawn position. (Prime letters will be used to represent the value of the variables for the fully drawn position. Prime letters will therefore be constants.)

$$dx/dt=d(D'-D)/dt=-dD/dt=V.$$

In like manner: $-dN/dt=v$ equal velocity of bow tip. But: $dD/dN=(dD/dt)/(dN/dt)=V/v=R$ where R is the ratio of the two velocities.

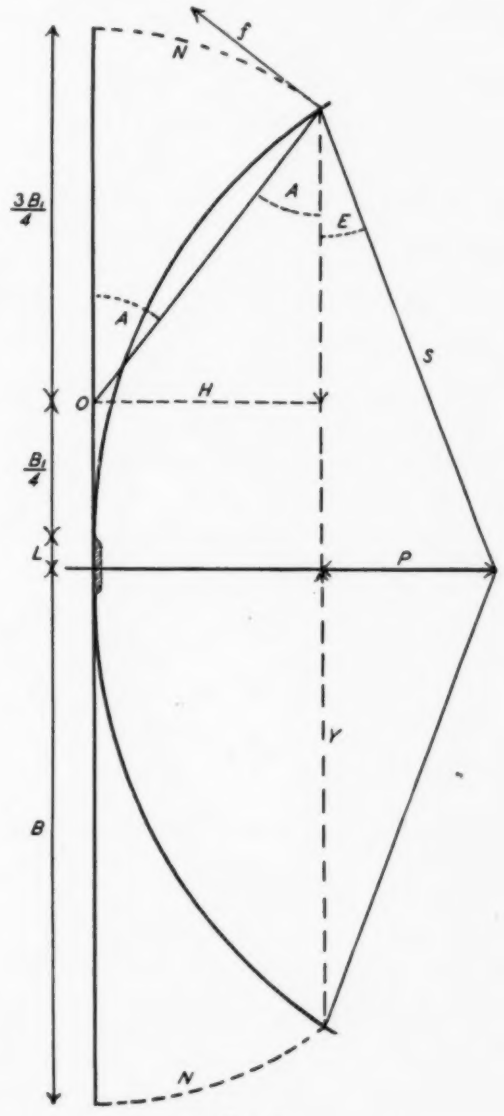

FIG. 1.

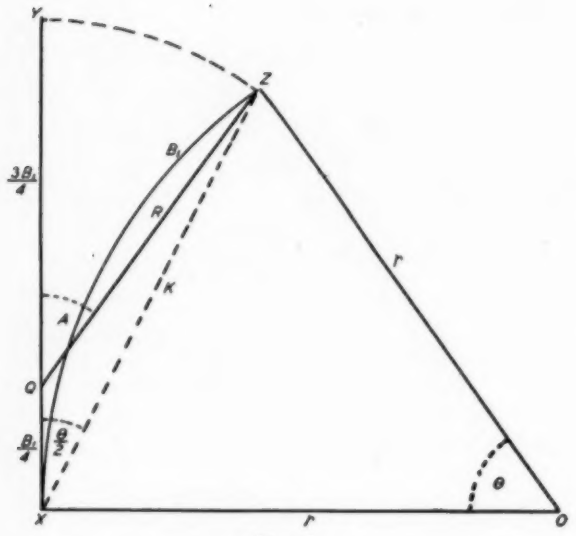

FIG. 2.

Also,

$$dD/dN=(dD/dA)(dA/dN).$$

But $D=H+P$. Therefore, substituting the values given for $D=H+P$ and differentiating

$$dD/dA=(3B_1/4)\cos A+(3B_1Y/4P)\sin A.$$

But $A=4N/3B_1$ and $dA/dN=4/3B_1$. Therefore:

$$dD/dN=V/v=R=\cos A+(Y/P)\sin A.$$

Let M equal mass of arrow, $m/2$ equal effective mass of each limb located at tip and W equal potential energy of both limbs. Since N is the displacement of each bow tip: $W'-W=C(A'N'-AN)$ is the loss in potential energy of both limbs from position of full draw to any other position.

Neglecting the dissipation in heat and sound, the gain in kinetic energy is equal to the loss in potential energy. Therefore:

$$MV^2/2+mv^2/2=C(A'N'-AN).$$

But $V=Rv$. Therefore:

$$v=[2C(A'N'-AN)/(MR^2+m)]^{\frac{1}{2}}.$$

But $N=3B_1A/4$. Therefore:

$$v=[3B_1C(A'^2-A^2)/2(MR^2+m)]^{\frac{1}{2}}.$$

And

$$V=Rv=R[3B_1C(A'^2-A^2)/2(MR^2+m)]^{\frac{1}{2}}.$$

Let **a** equal the acceleration of the bow tip and **A** the acceleration of the arrow. (Bold face type will be used to indicate dynamic values of variables.)

$$\mathbf{a}=dv/dt=(dv/dA)(dA/dt)$$
$$=(dv/dA)(dA/dN)(dN/dt).$$

But $dA/dN=4/3B_1$ and $dN/dt=-v$. Therefore: $\mathbf{a}=-(4v/3B_1)(dv/dA)$. Therefore by differentiation:

$$\mathbf{a}=2CA/(MR^2+m)$$
$$-2MRC(A'^2-A^2)Z/(MR^2+m)^2,$$

where

$$Z=\sin A-(Y/P)\cos A+(3B_1S^2/4P^3)\sin^2 A.$$

Also $\mathbf{A}=dV/dt=d(Rv)/dt=vdR/dt+Rdv/dt.$

$$dR/dt=(dR/dA)(dA/dt)$$
$$=(dR/dA)(dA/dN)(dN/dt)=-(4v/3B_1)dR/dt.$$

Therefore

$$\mathbf{A}=2CAR/(MR^2+m)$$
$$+2mC(A'^2-A^2)Z/(MR^2+m)^2.$$

The dynamical force **f** on the tip is: $\mathbf{f}=m\mathbf{a}/2$. The dynamical force **F** on the arrow is:

$$\mathbf{F}=M\mathbf{A}.$$

The author with a shooting machine which he designed for use in testing arrows. This machine was used in conjunction with a chronograph for measuring the velocity and acceleration of arrows. It is equipped with pneumatic release.

The dynamical tension in the string is:

$$\mathbf{T}=\mathbf{F}/2\sin E=S\mathbf{F}/2P=SM\mathbf{A}/2P$$
$$=(f-\mathbf{f})/\sin(A+E).$$

If the angle A is expressed in radians, the masses in pounds, the lengths in feet and the forces in poundals, the velocity and acceleration will be obtained in feet per second and feet per second per second, respectively.

It is more convenient, however, to use degrees for the angle A, inches for the lengths, pounds for the forces and grains for the masses. It is also desirable to have the constant C of such a value that $f=CA$ where f is expressed in pounds and A is expressed in degrees. In order to use the units as just explained and still obtain the velocities and accelerations in feet per second and feet per second per second, respectively, the above formulas become:

$$v=[492B_1C(A'^2-A^2)/(MR^2+m)]^{\frac{1}{2}},\qquad V=Rv,$$

$$\mathbf{a}=450{,}800CA/(MR^2+m)$$
$$-7{,}868MRC(A'^2-A^2)Z/(MR^2+m)^2,$$

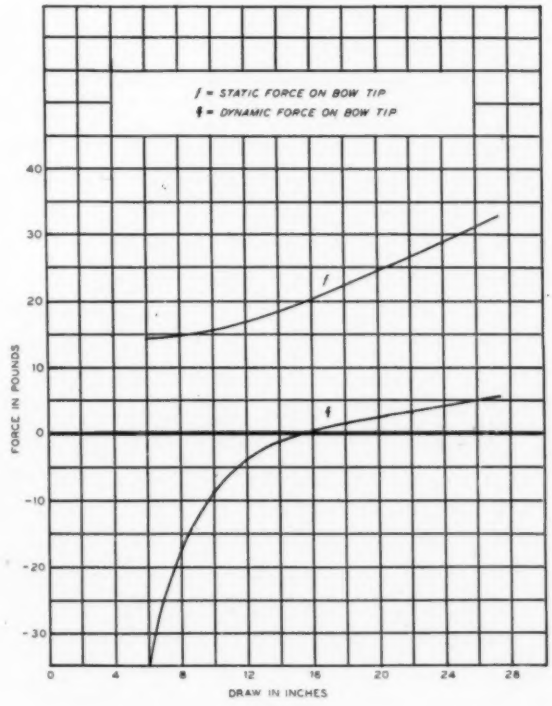

FIG. 3.

$$\mathbf{A}=450{,}800CAR/(MR^2+m)$$
$$+7{,}868mC(A'^2-A^2)Z/(MR^2+m)^2,$$

$$\mathbf{f}=2.22m\mathbf{a}\times10^{-6},\qquad \mathbf{F}=4.44M\mathbf{A}\times10^{-6},$$

$$\mathbf{T}=\mathbf{F}/2\sin E=S\mathbf{F}/2P=2.22SM\mathbf{A}\times10^{-6}/P$$
$$=(f-\mathbf{f})/\sin(A+E),$$

$$R=V/v=\cos A+(Y/P)\sin A.$$

In order to show how these formulas may be used, they are now applied to a six-foot bow which has an eight-inch rigid middle section ($L=4$ inches), and which has a six-inch bracing height ($H_0=6$ inches), and a draw $D=27.5$ inches.

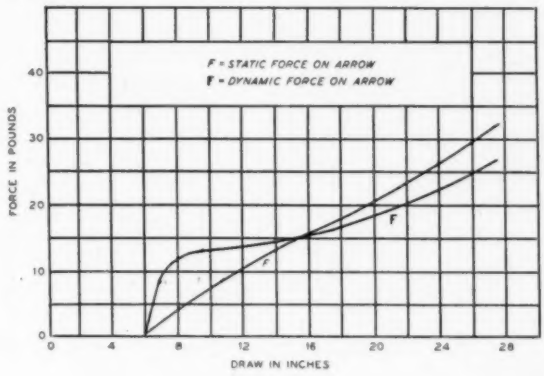

FIG. 4.

From the equation $H_0=(3B_1/4)\sin A_0$, $A_0=14°-29'$.

From the equation $S=(3B_1/4)\cos A_0+B_1/4+L$, $S=35.2373$ inches.

The active bending portion of each limb, $B_1=32$ inches.

Let each limb have a constant thickness, $t=0.6$ inch. Let the width at the dip equal $W=1.5$ inches. Let the width at any other point equal $w=Wx/B_1$, where x is the distance from the nock. This type bow is discussed in the January, 1932 issue of *Ye Sylvan Archer*. All sections of such a bow are equally stressed and the limbs bend in true arcs of circles.

It may be shown that the kinetic energy of a limb for this type bow $K=tWdB_1V^2/60$ where d is the density of the bow wood and V is the velocity of the nock as it passes through its neutral position.

If a bow limb having the same kinetic energy has all its mass $m/2$ concentrated at the nock, then:

$$mV^2/4=tWdB_1V^2/60$$

and

$$m=tWdB_1/15.$$

For m in grains, t, W, and B_1 in inches and d in pounds per cubic foot, this equation becomes:

$$m=0.270tWdB_1.$$

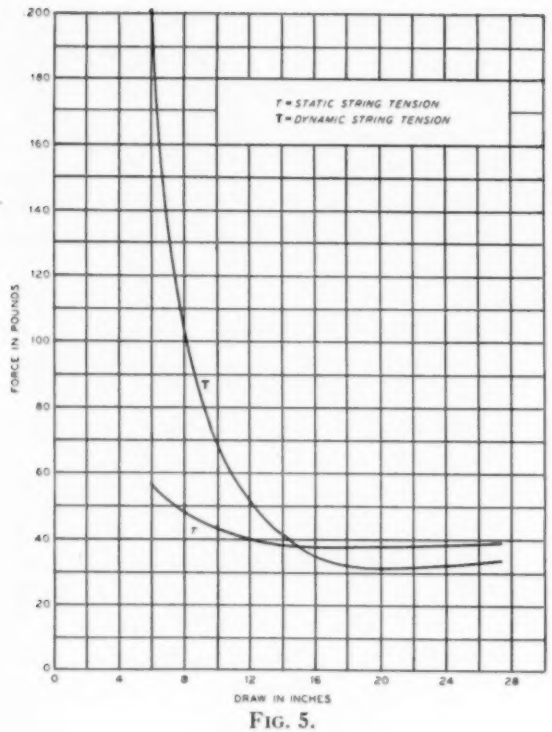

FIG. 5.

For yew wood, $d=43$. Therefore:

$$m=0.270\times.6\times1.5\times43\times32=334.4 \text{ grains.}$$

If the arrow weighs 350 grains and the string weighs 105 grains, M will equal 385 grains, because it has been shown in previous publications that approximately $\frac{1}{3}$ the weight of the string should be added to the weight of the arrow in order to obtain its effective mass M.

It is now necessary to determine the value of the constant C.

In the January, 1932 issue of *Ye Sylvan Archer*, it was shown that the deflection at the nock of this type bow $N=6fB_1^3/Wt^3Y$ where Y is Young's modulus for bending. But $N=B_1\pi A/240$ where A is expressed in degrees and

$$f=CA.$$

Therefore: $B_1\pi A/240=6CAB_1^3/Wt^3Y$

from which $C=Wt^3\pi Y/1440B_1^2$.

For yew, $Y=1.46\times10^6$.

Therefore $C=1.005$.

The formulas developed in this paper are now applied and the results plotted in graph form.

Fig. 3 shows the static and dynamic forces on the bow tip or nock, as a function of the draw. The curve marked f represents the force tangent to the path required to deflect the tip for all positions of draw. When the string is released with the arrow in place, most of the potential energy of the bow is used in accelerating the arrow so that there is initially only about five pounds of force effective in accelerating the bow tip. This dynamical force **f** becomes smaller as the arrow is being discharged. It reaches zero for a position corresponding to a draw of about 15.5 inches. This force then becomes negative. This means that the static force tending to move the tips forward is not as great as the retarding force resulting from the string tension. This difference is made up by the kinetic energy that has been acquired by the tips. The force which is tending to stop the tips is now being used to accelerate the arrow.

Fig. 4 shows the static and dynamic forces on the arrow. Here we see that at the moment of loose, the dynamic force **F** on the arrow is smaller than the static force F which was on the holding fingers. This is due to the fact that some force is required to start the limbs in motion. As the arrow moves forward, the force on it decreases as shown by the curve **F**. The

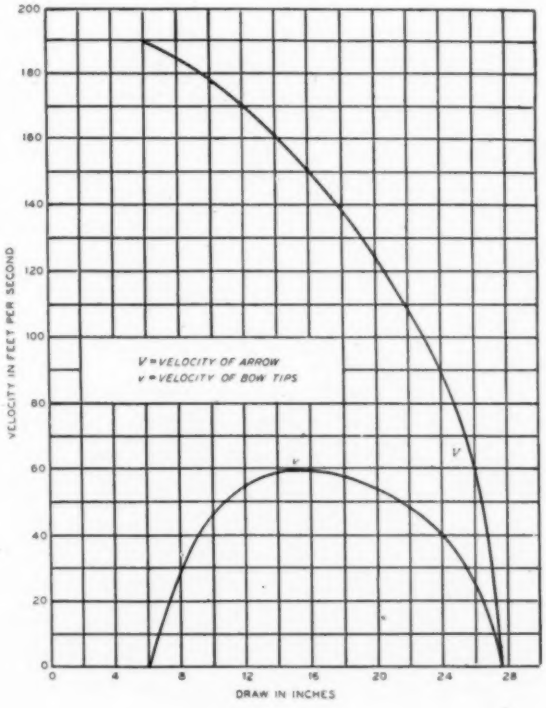

FIG. 6.

dynamical force **F** does not decrease as rapidly as the static force *F*. This is due to the kinetic energy of the tips. This explains why massive bow tips do not reduce the velocity of the arrow by a very great amount. The energy required to put the tips in motion is given to the arrow at a later time. However, this increase in the force **F** due to the kinetic energy of the limbs may not be desirable from the standpoint of accuracy.

The static string tension *T* is given in Fig. 5. In the same graph we find that as the arrow is loosed, the dynamical tension **T** drops slightly but soon begins to increase. Were it not for the fact that the string may stretch slightly, this tension would break the string. Heavy tips increase this tension.

Fig. 6 gives the velocity of the arrow and the velocity of the bow tips as a function of the draw. We see that the velocity of the bow tips becomes a maximum corresponding to a draw of about 15 inches. It then becomes smaller and smaller, reaching zero at the time the string reaches its neutral position. Because of stretching of the string this zero velocity is reached slightly later than shown. However, in this case the tips actually reverse their direction and move back again. A heavy string accentuates this effect. The velocity of the arrow increases throughout the time of discharge. However, for light arrows and heavy tips, the rate of increase may become much less during the middle portion of the discharge and then increase quite sharply near the end.

Fig. 7 shows the accelerations of the arrow and of the bow tips as functions of the draw. The acceleration of the bow tips becomes smaller and smaller until it reaches zero at a position corresponding to a draw of about 15.5 inches. The acceleration then becomes negative. This means that the tips are slowing up. The acceleration of the arrow decreases for a short distance and then the decrease becomes less rapid. For very light arrows it may increase again. The shorter the bow, the more the reflex, the heavier the tips and the lower the stringing height, the more pronounced is this effect. That is why so many archers condemn all these conditions where accuracy is desired. A bow having a permanent set will reduce this effect. Heavy arrows will also prevent the acceleration from increasing near the close of the discharge. A whip-ended bow will also reduce the acceleration near the end of the discharge. This explains why a whip-ended bow does not kick.

The computations given here are for a bow having an efficiency of 100 percent. If the string

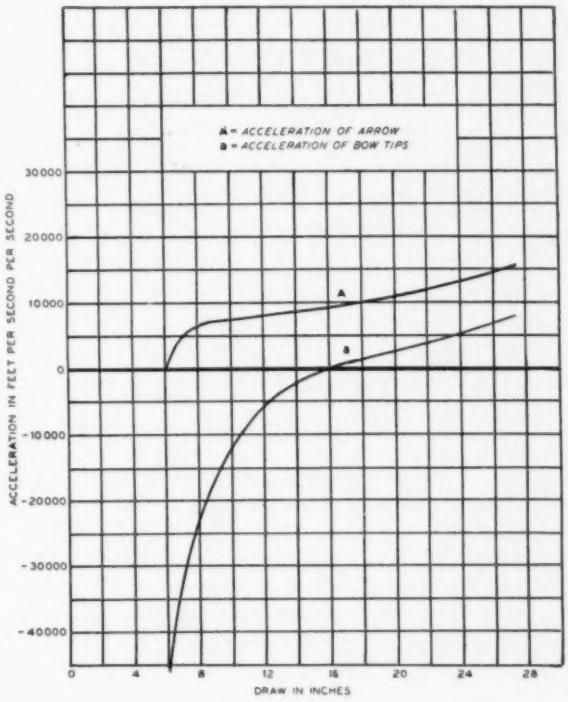

FIG. 7.

is taken into consideration as in the above computations, bows may be built that have an efficiency of 93 percent. An Osage bow of the type described above was built by Dr. P. E. Klopsteg which, after taking the string into consideration, had an efficiency of 93 percent. Most bows have much lower efficiencies. If the bow wood is good and no part is stressed beyond its elastic limit, the efficiency will be lowered only by the fact that the kinetic energy of the limbs is not all transferred to the arrow. The loss within the wood and in air resistance is usually negligible. If, in making the computations, the final velocity of the arrow is taken for a position about 2 inches in advance of the bracing height, a value will be obtained that compares well with experimental results. In other words, in practice all of the kinetic energy of the tips is not converted into arrow velocity as assumed in the computations.

Here and There

National Research Council Fellowships

¶ National Research Council Fellowships in physics for 1937 have been awarded to the following:

	Proposed Institute for Study
Emil John Konopinski	Cornell University
Alfred Otto Carl Nier	Harvard University
Hubert José Yearian	Harvard University
Erwin Rudolf Gaerttner	California Institute of Technology
Jesse Leonard Greenstein	Yerkes Observatory
William Conyers Herring	Massachusetts Institute of Technology
William Rudolph Kanne	University of Wisconsin
Alexander S. Langsdorf, Jr.	University of California
John Reginald Richardson	Princeton University
Leonard Isaac Schiff	University of California

★

$1000 American Chemical Society Prize

¶ The $1000 prize of the American Chemical Society for 1937 has been awarded to Dr. E. Bright Wilson, Jr., assistant professor of chemistry in Harvard University. The award, bestowed annually upon a scientist under thirty-one years of age and of unusual promise, goes to Dr. Wilson, who is twenty-nine, for his experimental work in physical chemistry.

The prize will be formally presented at the ninety-fourth meeting of the society to be held in Rochester, N. Y., September 6 to 10. Maintained by Dr. A. C. Langmuir of Hastings-on-Hudson, N. Y., and his brother, Dr. Irving Langmuir, Nobel prize winner in chemistry, it provides for "recognition of the accomplishment in North America of outstanding research in pure chemistry by a young man or woman, preferably working in a college or university."

★

Science on the Radio

¶ A description of the grinding of the world's largest telescopic mirror and a demonstration of the operation of gigantic dynamos which create artificial lightning were highlights on a broadcast from the California Institute of Technology, at Pasadena, on Thursday, April 29, from 8:00 to 8:30 P.M., EDST, over the Blue Network of the National Broadcasting Company.

Dr. Carl D. Anderson, winner of the Nobel prize, was among the experts who gave listeners actual demonstrations of tests conducted in wind tunnels, high tension laboratories, and the optical laboratory. From the optical laboratory, under the supervision of Dr. Max Mason, listeners heard the actual grinding of the world's largest telescopic mirror, a 200-inch glass. Professor Clark Millikan, head of the aeronautical division, described the aeronautical laboratory.

Physics Teachers of Western Pennsylvania and Environs

¶ The Association of Physics Teachers of Western Pennsylvania and Environs met May 15, at St. Vincent College near Latrobe, Pa. The program was as follows:

Welcome. Rt. Rev. Alfred Koch, O.S.B., *Archabbot.*
What is considered an adequate lecture in physics? W. N. St. Peter, *University of Pittsburgh.*
Apparatus for automatically demonstrating the appearance of various transparent substances in plane polarized light (with demonstration). J. B. Nathanson and C. W. Prine, *Carnegie Institute of Technology.*
Supply and maintenance of equipment. Father Bernard, O.S.B., *St. Vincent College.*
The reflecting telescope as an undergraduate project. W. P. Cunningham, *State Teachers College, California, Pa.*
Some characteristics of good definitions. A. G. Worthinc, *University of Pittsburgh.*
Physics in industry. David J. Giles, *Works Manager, Latrobe Electric Steel Company.*
Physics and steel. Leonard Grimshaw, *Latrobe Electric Steel Company.*
Energy loss in hysteresis, in terms of the recent ferro-magnetic theory. G. A. Scott, *University of Pittsburgh.*
Laboratory experiments. F. A. Molby, *West Virginia University.*

★

Award of the Acheson Medal

¶ The Electrochemical Society has just awarded the Acheson Medal and $1000 prize to Dr. Frederick Mark Becket, president of Union Carbide and Carbon Research Laboratories, Inc., and vice president of the Electro Metallurgical Company and the Union Carbide Company, for his outstanding contributions to electrometallurgy. The highly developed alloy steel industry of today, with its stainless alloys, its high speed tools, and high temperature resistant products used in the electrical industry, owes much of its rapid spread to Becket's inventions.

★

Fellowship in Electrochemistry

¶ The Electrochemical Society has just awarded the ninth Weston Fellowship of $1000.00 to Mr. Garth L. Putnam, of Seattle, Washington. Mr. Putnam will continue his research at Columbia University, investigating methods of efficiently depositing alloys of active metals from aqueous solutions. The work will be done under the direction of Professor Colin G. Fink.

The Fellowship was founded by Dr. Edward Weston, a pioneer in electrochemistry.

★

Behind the Scenes in Industrial Europe

¶ The European laboratory tour party of the National Research Council sailed May 19 on the S.S. Normandy. The party numbered about forty. They will spend two weeks visiting laboratories in England and about two weeks each in France and Germany.

University Department Changes

¶ In recognition of steadily expanding scientific knowledge and the prospect of increasingly important developments in industry, the Massachusetts Institute of Technology has announced the division of its Department of Mining and Metallurgy into two distinct fields of professional education.

Beginning next autumn the Department of Mining Engineering will be administered as a separate department under the direction of Professor W. Spencer Hutchinson, who has long been head of the courses in mining and metallurgy. His department will also include the course in petroleum production. The new Department of Metallurgy will be directed by Dr. Robert S. Williams, for many years professor of physical metallurgy, and will include the course in ceramics.

The new Department of Metallurgy will give special attention to physical and process metallurgy, as well as metallurgical production, the latter being concerned primarily with the economical and statistical functions of the industry. The two former involve application of basic and scientific principles to the production and treatment of metals.

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¶ At the College of Engineering of the Carnegie Institute of Technology, courses in chemical engineering will be administered next autumn under two separate departments instead of under one as at present. Dr. Thomas R. Alexander, associate professor of inorganic chemistry, will become acting head of the department of chemistry, and Dr. Warren L. McCabe, professor of chemical engineering, will be head of the department of chemical engineering.

★

The Homes of Tomorrow

¶ A conference on housing was held at the Massachusetts Institute of Technology in Cambridge, June 7. The program arranged for this meeting was as follows:

Vannevar Bush, Vice President and Dean of Engineering, M.I.T., presiding and summing up at the end.
Sir Raymond Unwin of London, England, world-famous architect and town planner, on "Better Homes and Neighborhoods for All—The Social and Economic Aspects of Shelter."
John Ely Burchard, Vice President of Bemis Industries, housing researcher, on "How Better Homes Will be Built—The Question Mark of Prefabrication: New Materials, New Methods of Design."
Ernest J. Bohn, sometime President, National Association of Housing Officials, consultant on low-cost housing, on "Government Intervention in Housing—What Governmental Agencies Have Done and Might Do."
Robert D. Kohn, distinguished New York architect, former Director, Housing Division of Public Works Administration, on "The Future of Housing—Some Possible Ways Leading to Better Housing."

★

¶ Dr. William H. Doherty, of the Bell Telephone Laboratories, has been awarded by the Institute of Radio Engineers the Morris Liebman Memorial Prize for 1937, in recognition of his work in the field of radiofrequency power amplifiers. The presentation was made at the annual dinner of the institute in New York on May 12.

Science, May 21, 1937

New Appointments and Promotions

¶ At Massachusetts Institute of Technology:

John Wulff from assistant professor to associate professor.
Wayne B. Nottingham from assistant professor to associate professor.
Nathaniel H. Frank from assistant professor to associate professor.

At Princeton University:

Gaylord P. Harnwell from assistant professor to associate professor.

At Cornell University:

Hans A. Bethe from assistant professor to professor.
H. E. Howe from assistant professor to professor.
R. F. Bacher from instructor to assistant professor.

★

¶ The degree of doctor of laws will be conferred in June by the University of Glasgow on Dr. Max Planck, emeritus professor of theoretical physics at the University of Berlin.

★

¶ Professor Linus Pauling has been appointed to succeed the late Professor Arthur A. Noyes as director of the Gates Chemical Laboratories of the California Institute of Technology. He has also been appointed George Fisher Baker lecturer in chemistry at Cornell University for the first term of the year 1937–38.

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¶ Professor W. L. Bragg, Langworthy professor of physics in the University of Manchester, has been appointed director of the British National Physical Laboratory.

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¶ Dr. Victor Hicks has joined the staff of the Westinghouse X-Ray Company, Inc., at Long Island City, New York.

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Journal of Applied Physics

For July

Science, Invention and Society, *by* WALDEMAR KAEMPFFERT

Physics in the Paper Industry, *by* E. W. SAMSON

Physics of Air Hygiene, *by* H. B. MELLER AND F. F. RUPERT

Physics of Subsidence and Ground Movement in Coal Mines, *by* H. P. GREENWALD

Trilinear Coordinates, *by* HOWARD AIKEN

Impregnation Studies with Colloidal Graphite, *by* BERNARD H. PORTER

Contributed Research Papers on Mechanisms for the Starting Potentials of Corona Discharge, Penetration Equipment for Estimating Viscosities, and Elastic Stretch and the Infrared Spectrum of Rubber.

¶ In response to a request for an opinion on "The Place of Science in National Planning," by Senator Robert J. Bulkley, chairman of the United States Senate Committee on Manufactures, the American Institute sponsored a discussion at the Chemists' Club, New York City, on April 22. Dr. Willis R. Whitney, of the General Electric Company, Waldemar Kaempffert, science editor of The New York Times, Dr. George Baehr, trustee of the New York Academy of Medicine, Julius H. Barnes, formerly chairman of the U. S. Chamber of Commerce, and others were present and led the discussion.

Science, April 23, 1937

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Announcements

¶ Sir William Bragg will deliver the forty-third James Forrest lecture before the Institution of Civil Engineers on April 27, at 6 P.M. The title of Sir William's lecture will be "The Crystal and the Engineer."

¶ Professor J. Chadwick will deliver the twenty-eighth Kelvin lecture before the Institution of Electrical Engineers on April 22 at 6 P.M. The subject of the lecture will be "Elementary Particles of Matter."

Nature, April 17, 1937

★

Electrochemical Society Elects New Officers

¶ At the annual meeting of the Electrochemical Society held at Philadelphia, Pennsylvania, the following new officers were elected:

President; WILLIAM G. HARVEY, Aluminum Company of America, Cleveland, Ohio.
Vice Presidents; L. D. VORCE, New York City
R. L. BALDWIN, Niagara Falls, New York
O. W. STOREY, Chicago, Illinois
Managers; W. W. WINSHIP, New York City
E. M. BAKER, Ann Arbor, Mich.
S. SWANN, JR., Urbana, Illinois
Treasurer; ROBERT M. BURNS, 463 West Street, New York City.
Secretary; COLIN G. FINK, Columbia University, New York City.

★

A. S. H. V. E. Semi-Annual Meeting

¶ The American Society of Heating and Ventilating Engineers will hold their semi-annual meeting at the New Ocean House, Swampscott, Mass., June 24–26, 1937. The following technical papers are scheduled:

Adaptability of pre-cooling coils to air washer systems. JOHN EVERETTS, JR.
Degree days for summer air-conditioning. A. D.-MARSTONS
Study of summer cooling in the research residence using water at temperatures of 52° and 46°F. A. P. KRATZ, S. KONZO AND E. L. BRODERICK
Cooling load analysis of a bank building. JOHN JAMES
Ventilation requirements, Part II. C. P. YAGLOU AND W. N. WITHERIDGE
Performance of fin tube units for air cooling and dehumidifying. G. L. TUVE AND C. A. MCKEEMAN
Cooling and heating rates of a room with different types of steam radiators and convectors. A. P. KRATZ, M. K. FAHNESTOCK AND E. L. BRODERICK
Report of A.S.H.V.E.-A.S.R.E. Committee on National Standards for Air-Conditioning Applications. L. A. HARDING, *Chairman*
Simplified method of testing and rating air cleaning devices. ARTHUR NUTTING
Direct reading effective temperature indicator. J. R. PARSONS
Infiltration effects on heating Grant building. F. C. HOUGHTEN

¶ At the annual meeting of the American Academy of Arts and Sciences, held on May 12 at its house, 28 Newbury Street, Boston, Massachusetts, it was voted to award the Rumford Medal to William Weber Coblentz, physicist of the National Bureau of Standards, Washington, D. C., in recognition of his investigations in heat and light, including the physical study of the firefly, photoelectric properties of materials, the radiation of stars and pioneer work on the temperatures of the planets.

Science, May 21, 1937

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¶ Dr. Thomas C. Poulter, director of the Armour Institute of Technology, Chicago, who was second in command of the Byrd Antarctic expedition of 1933–35, was presented on April 27 with a special gold medal by the National Geographic Society. Dr. Gilbert Grosvenor, president, made the presentation in the presence of Rear Admiral Richard E. Byrd. On the Byrd expedition Dr. Poulter's work included geophysical investigations, studies of ice conditions, magnetic soundings and observation of meteors and auroral phenomena.

★

¶ Science Service announces that the Massachusetts Institute of Technology will build a ten-foot diameter tunnel that will simulate conditions at 35,000 feet altitudes and speeds of 450 miles an hour. Similar equipment is already being built at Langley Field by the National Advisory Committee for Aeronautics. Both wind pressure tunnels follow the general plan and design of the wind tunnel of the Reich Air Ministry at the University of Goettingen at Hannover, Germany. The rarefied atmosphere and accompanying low air pressures encountered in the sub-stratosphere, where aviation now would like to do its flying, render inaccurate some experimental findings obtained in previous wind tunnels at ordinary atmospheric pressure.

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¶ The Research Foundation of Armour Institute of Technology announces the establishment of the Eli Wishnick Fellowship to promote fundamental research in the field of extremely high pressure. This fellowship carries a stipend of Seven Hundred Fifty Dollars a year. The Fellow may work for either an M.S. or a Ph.D., although preference will be given to applicants having a master's degree, and intending to work for a doctorate. The appointment will be made for a two-year period beginning September, 1937. Application should be made to Thomas C. Poulter, executive director of the Research Foundation of Armour Institute of Technology, Thirty-three Hundred Federal Street, Chicago, Illinois.

★

Necrology

¶ Alexander Howard McConnell, founder, and chairman of the board of directors of Central Scientific Company, died at Albuquerque, New Mexico on May 4, 1937, in his sixty-first year, while en route to Chicago from California.

Among New Books

Electrical Engineering in Radiology. L. G. H. SARSFIELD. Pp. 284+xiv, Figs. 199. Instruments Publishing Company, Pittsburgh, 1936. Price $6.00.

This is a fairly rapid survey of x-ray apparatus from the point of view of the design engineer. The contents are a mixture of design data, circuit information, practical techniques in operation and control, and general descriptions of commercial apparatus which should prove useful to those wishing to be oriented in some of the problems, methods, and apparatus of the conventional x-ray field. There are chapters on the design of x-ray potential and filament transformers, on mechanical rectifiers and electronic-tube rectifying circuits, on x-ray tube current and voltage measurements, high voltage cables, voltage and current control apparatus, and on equipment layout. The chapter surveying standard x-ray tubes includes mainly general descriptions and illustrations of commercial tubes. The book concludes with brief chapters on maintenance of x-ray equipment, special equipments, and rules of electrical safety.

Electrical Engineering in Radiology may prove of interest to roentgenologists and technicians in charge of x-ray equipment, to prospective purchasers of equipment, and to students. It is hardly sufficiently detailed or technical for reference use by engineers and physicists already in the field. The author has been connected with the electrical development of radiological equipment at the Woolwich Arsenal in England, and it is therefore quite natural that the emphasis should be on English and continental apparatus and procedures. It will perhaps be regretted by many that the subject matter has been restricted to the power supply, circuit, and practical engineering aspects of x-ray production, and that such important matters as x-ray phenomena, intensity and quality determinations, filtration, electronics of x-ray tubes, and x-ray vacuum technique, which might quite properly come under engineering in radiology, are not considered. The material covered, however, is well written with many illustrations, and supplies much practical information.

JOHN G. TRUMP
Massachusetts Institute of Technology

Dynamics, Part II. A. S. RAMSEY. Pp. 344+xi, illustrated. Cambridge University Press, England, 1937. Price $4.50.

This book is a continuation of an earlier volume under the same title by the same author and deals with the more advanced parts of dynamics. Both are written in true Cambridge style, the first volume being intended for "first year honours' students" and the second part having for its object a three years' "honours course" and the famous "Tripos examinations." The author is president of Magdalene College in Cambridge, and in past years has prepared for publication a *Treatise on Dynamics* by Dr. Besant, from which the present book has drawn some material.

Coming from these surroundings it is considerably more difficult and thorough, and contains much more subject matter than the usual textbook of American colleges. Comparing its scope and difficulty with other English books on the subject, it seems to stand somewhere between the works of Lamb and Routh, and, like those classics, contains a large number of problems and worked-out examples.

The book opens with some special topics in particle dynamics, including theory of orbits and rotating tubes, followed by a short chapter on motions of chains. After a brief exposition of kinematics various motions of particles in three dimensions are taken up. This is followed by two chapters on two-dimensional motion of rigid bodies. Next three-dimensional motions, including impulsive motions, are treated extensively by means of Newton's equations.

The next large chapter on generalized coordinates brings Lagrange's and Hamilton's theorems with many examples. These methods are applied in the following two chapters to small oscillations and gyroscopic motions. An appendix deals with vector analysis and its applications, of which the author states that "it is hardly possible to ignore the present day fashion of using vectors in dynamics, but I doubt whether it is a fashion which has come to stay and I am unwilling to require all readers of this book to use vectors."

Summarizing, it may be said that the book is clearly written, contains a large number of good problems, and, with its first part, will be useful for those who want to study the subject thoroughly, but who hesitate to take up the more extended works of Routh or Rayleigh.

J. P. DEN HARTOG
Harvard University

IX^e Congrès International de Photographie. Pp. 852. Published by L. P. Clerc, La Revue d'Optique Théorique et Instrumentale, Paris, 1936.

Representing a cross section of the photographic art at the time (1935), the Proceedings of the Ninth International Congress, held in Paris, are worthy of perusal by all who use photography as a scientific implement. American readers will be interested in this record of the first public European exhibition of Kodachrome film and of photographs taken with the aid of Pola screens. Three papers are concerned with the sensitometry of bichromated gelatin, a subject on which little has been published. Theorists will be pleased to note the revival of interest in

the theory of photography that has taken place since the Dresden Congress in 1931.

Of the papers having the most practical implications, it was invidious to select. As an indication of the scope of these papers, there might be mentioned the regeneration of M-Q developers (p. 219), the making of Schumann plates totally devoid of gelatin (p. 309), a sensitometric study of reducers (p. 365), the use of ethyl bromacetate in the developer to produce bluish tones on chloride papers (p. 388), a pessimistic study of fine-grain developers (p. 396), the application of the Herschel effect to the printing of negatives having excessive contrast (p. 438), an inertialess light valve for sound recording based on the Kerr effect (p. 717), a photographic method of reading a legend that has been obliterated with ink of a different color (p. 727), and a survey of the problem of photographing paintings in monochrome (p. 768). Those who wish to amuse themselves by sensitizing their own paper will be interested in formulas given on pages 372–386. Unusual applications of photography in paleontology are described (p. 737), as well as x-ray motion pictures of the vocal cords, with sound accompaniment (p. 753), "*qui provoqua . . . les rires de l'assemblée.*"

Unlike the Dresden Congress, which standardized the type of light source to be used in sensitometry, the Paris Congress effected no standardization. For speed measurement, the German DIN (Deutsche Industrie Norm) system (unfortunately not described in the Proceedings) was the leading contender, but it was rejected, pending further investigation, because the required development conditions result in too high a gamma. The counter proposals of the other delegations and the arguments for and against the DIN system give the reader an idea of the extent to which a single quantity will indicate the sensitivity of a material in practical use.

FRED H. PERRIN
Eastman Kodak Company

The Analysis of Engineering Structures. A. J. S. PIPPARD AND J. F. BAKER. Pp. 553+ix, illustrated. Longmans Green, New York, 1937. Price $9.00.

Both authors are professors of English engineering schools and both have been taking an active part in the work of the Steel Structures Research Committee of the Department of Scientific and Industrial Research. The book contains not only material which is usually given to students in courses on theory of structures, but also some new material representing the research work of the authors. The content of the book is briefly as follows:

The first three introductory chapters contain a review of methods of analysis of simple trusses and methods of calculating stresses and deflections in straight and curved beams. The 4th chapter deals with general theorems concerning the analysis of statically indeterminate structures. Chapters 5 and 6 give applications of these theorems in calculating deflections and stresses in redundant frames. Chapters 7 and 8 deal with struts and beam submitted to the simultaneous action of axial and lateral loads. Various column formulae are also discussed in these chapters. In chapters 9, 10, and 11 systems with rigid joints are considered and the methods of calculating secondary stresses in trusses are discussed. Chapters 12 and 13 deal with elastic arches and with reinforced concrete beams. In chapters 14 and 15 the influence lines for statically determinate and for redundant frames are shown, and the impact effect on bridges is discussed. From the remaining portion of the book we will mention chapter 19, treating the design of steel framed structures, in which the preliminary investigations by the Steel Structures Research Committee are discussed. At the end there are six more chapters briefly discussing suspension bridges, curved beams loaded normally to the plane of curvature, flat slabs, gravity dams, earth pressure, and the mechanical methods of stress analysis.

In several places new material can be found which usually is not given in textbooks. In chapter 4, for instance, a method is developed for design of redundant frames which makes possible the determination of cross-sectional area of all members so as to make the stresses equal to the allowable stress. This method is more direct than the usual trial and error method. In chapters 9 and 10 the methods of successive approximations are discussed at some length. Essentially, the work of American engineers is given in the book and the reader may obtain an impression that these methods represent very recent developments. In fact, the originator of these methods was O. Mohr, who used such a method for calculating secondary stresses.[1] The same method was applied to numerical examples in Waddell's book[2] and in this way great simplification in calculating secondary stresses was obtained. The adaptation of the method to the analysis of rigid frame structures such as we encounter in the design of office buildings was made by K. A. Čališev.[3] He developed, also, a very useful method of analysis of Vierendeel trusses.[4] The recent development of the method of successive approximation is similar in principle to the solution used by Čališev. In chapter 14 new material regarding impact effects on bridges is given and vibration of bridges produced by balance-weights is discussed. The new material in chapter 19 has already been mentioned.

The authors refer principally to the literature in the English language and do not always give proper reference to the work done on the Continent. In discussing bending of beams, for instance, they refer to W. H. Macaulay's paper, although the method was originated with Saint Venant. In presenting the generalized theorem of three moments, they mention the paper by Arthur Berry, in 1916, although similar equations were obtained by several authors; indeed, in 1916 these equations were already in textbooks. Their discussion of beams curved in plane gives reference to the work by A. H. Gibson and E. G. Ritchie, although problems of this kind were discussed a long time ago by Saint Venant. In fact, there exists broad literature on this subject, principally in the German language.

It is the opinion of the reviewer that the book is not complete enough to be used as a text in American engineering schools. There is very little given about methods of analysis of trusses and hinged three-dimensional systems. Practically nothing is said about the design of Voussoir

arches. All recent work on the choice of the center line of arches, the effect of compressive forces on bending of arches, the elastic stability of arches are not mentioned. The discussion of thin walled structures such as are encountered in airplane design is also omitted. At the same time, the book contains some new material and may be of interest to engineers who would like to know about the new developments in the theory of structures as accomplished in England.

S. TIMOSHENKO
Stanford University

[1] See Mohr's book, *Abhandlungen* (1906), p. 429.
[2] J. A. L. Waddell, *Bridge Engineering* (1916).
[3] Tehnički List, Zagreb, 1922 and 1923. See also the recent paper of the same author in *Memoirs of the International Association for Bridge and Structural Engineering*, Vol. 4 (1936).
[4] See K. A. Čališev, "Die Berechnung der Rahmentrager," Der Bauingenieu, 1922. See also L. C. Maugh, *Memoirs of the International Association for Bridge and Structural Engineering*, Vol. 3 (1935).

Katalytische Umsetzungen. W. FRANKENBURGER. Pp. 444+xi. Akademische Verlagsgesellschaft, Leipzig, 1937. Price, bound, RM 36.

Lehrbuch der Hockfrequenztechnik. F. VILBIG. Pp. 775+xx, Figs. 801. Akademische Verlagsgesellschaft, Leipzig, 1937. Price, bound, RM 32.80.

The Logical Structure of Science. A. CORNELIUS BENJAMIN. Pp. 344. Kegan Paul, Trench, Trubner & Co., Ltd., Broadway House, London, 1936. Price 10s. 6d.

Dielectric Phenomena in High Voltage Cables. D. M. ROBINSON. Pp. 173+xii, Figs. 96. Instruments Publishing Company, Pittsburgh, Pa., 1936. Price $5.00.

Contributors to this Issue

Dr. A. Nadai received his early training at the Technische Hochschule, Zurich, Switzerland. He received the degree of Doctor of Engineering at the Technical University of Berlin in 1911 and was Professor of Applied Mechanics at the University of Goettingen until 1928. In 1929 he joined the Research Laboratories of Westinghouse. He is the author of several books on applied mechanics. During the summer of 1937 he will give a course on the theory of the creep of metals at Massachusetts Institute of Technology, Cambridge.

Professor Hermon W. Farwell was born at Keene, New Hampshire, in 1879. In 1902 he received his A.B., and in 1906 his A.M. degrees at Dartmouth College. He has been successively instructor of physics at Dartmouth College, assistant, tutor, instructor, assistant professor, associate professor, and at present he is professor of physics, at Columbia University.

Dr. Morris Muskat received his A.B. and A.M. degrees from Ohio State University in 1926 and was a fellow at the California Institute of Technology from 1927–29, taking his Ph.D. degree at that institute in 1929. From 1926 to 1927 he was an instructor in physics and chemistry at Bowling Green State Normal College, Ohio. Since 1929 he has been a theoretical physicist at the Gulf Research & Development Co., Pittsburgh, Pa. He is now in charge of the physics division in the Gulf laboratory.

Dr. M. Alden Countryman was born at St. Paul, Minnesota, in 1903 and graduated with a B.S. in electrical engineering from the University of Minnesota in 1925. He received his M.S. in physics and Ph.D. in applied physics in 1933 and 1936 at the Iowa State College. From 1926 to 1932 he was employed as sales engineer and office engineer by the Northern States Power Company at Sioux Falls, South Dakota, and is now an instructor in the engineering department at Iowa State College.

Dr. Lloyd P. Smith was born at Reno, Nevada, in 1903. He received his bachelor's degree in electrical engineering at the University of Nevada in 1925, was a Coffin Fellow during 1926 and 1927, and took his Ph.D. at Cornell University in 1930. He was a National Research Council fellow at the California Institute of Technology, 1930–31 and an International Research Fellow at Munich, 1931–32. He spent one year in the General Electric Research laboratory and has been an instructor and assistant professor of physics and is now professor of physics at Cornell University.

Contributed Original Research

Double Refraction in Colloidal Solutions

H. W. Farwell
Columbia University, New York, N. Y.
(Received March 31, 1937)

THE phenomenon of double refraction during the flow of certain colloidal solutions is well known.[1] In these solutions it is fairly clear that the colloidal particles must not be considered as amorphous spheres but as minute crystals which are themselves doubly refracting, and that the orientation in the stream is evidence of the more or less needle-like form of these crystals.[2] The first evidence of the sort was obtained by Kundt[3] who was investigating double refraction in liquids by the rotating cylinder method of Maxwell.[4] Kundt's results indicated a differentiation between double refraction which is proportional to rate of slip and double refraction which is dependent upon orientation of long colloidal particles. Whether or not the two are produced by the same cause, an actual shearing stress, it has seemed to be desirable to ascertain whether in reality there were two effects or only one.

If the effect is single, then double refraction is definitely related to rate of slip, and becomes zero when the rate of slip is zero. As a first attempt to secure the proper conditions a thin stream of a vanadium pentoxide solution was forced through a narrow slot in the bottom of a reservoir over which air was admitted under pressure. The stream passed vertically downward with its plane parallel to two crossed Polaroid plates about 6.5 cm in diameter set with principal directions at 45° to the direction of motion of the stream.

While there is shearing during the passage in the slot, once the liquid has passed beyond the orifice, there can be left only such small shears as are produced by surface tension and by contact with the air. The effect of contact with the air is to produce a possible slip in the direction of motion. If the plane of slip is parallel to the direction of propagation of light, then the principal stresses are normal to the direction of propagation of light and at 45° to the vertical, that is, parallel to the directions of polarizer and analyzer. Hence this cannot change the intensity of the light transmitted. If the plane of slip is perpendicular to the direction of propagation, the principal stresses are in the vertical plane parallel to the direction of the light. Hence a small effect may be due to this slip. The effect of surface tension is to contract the stream and is therefore tending to produce slip in the horizontal direction, the effect of which can be analyzed in a similar way.

Actually the entire stream in the field of the Polaroid plates showed a brilliant fairly uniform color. The intensity was much greater than that observed when the plates were set with polarizing directions vertical and horizontal. The conclusion is that in that stream the rod-shaped particles were largely oriented with the long dimension along the direction of the stream, and held that position at least during the passage through the field. However, there is not a complete elimination of the effect of slip.

A second attempt to find double refraction in a fluid with rate of slip zero was made by studying the effect of introducing a cell containing a V_2O_5 sol or a benzopurpurin sol between the

[1] E.g., Freundlich, *Elements of Colloidal Chemistry*, p. 134.
[2] A general summary may be found in Coker and Filon, *Photoelasticity*.
[3] Kundt, Ann. d. Physik **13**, 110–133 (1881).
[4] Maxwell, *Collected Papers*, Vol. II, p. 379.

crossed Polaroid plates. (The cell was about 5 cm high, 3 cm wide, and 1 cm thick.) The double refraction in the cell is very plain if there is any flow in the cell, and some of the patterns obtainable are quite remarkable. (Several are shown to indicate the type of phenomenon observed.) The patterns were often changing rapidly, and even small temperature differences were quite sufficient to start convection currents in which long particles were given other than a random orientation.

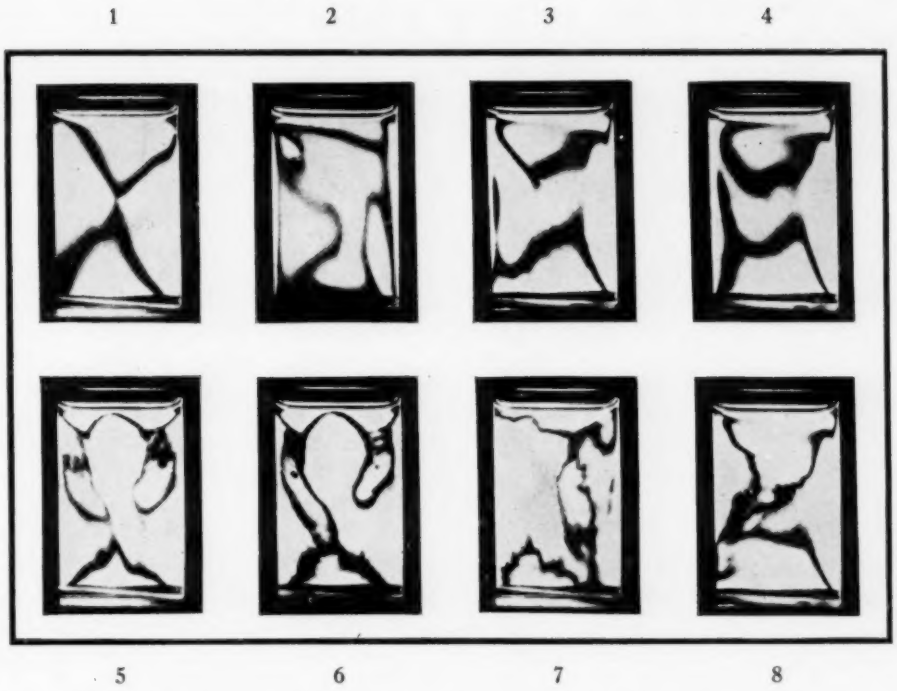


FIG. 1. Cell after standing overnight. No water bath. FIG. 2. Lower right side of cell is being warmed by a jet of air on the outside. FIG. 3. Air jet discontinued. FIG. 4. Several minutes after No. 3. FIG. 5. Cell oscillated slightly for a few seconds. FIG. 6. Another view after oscillation. FIG. 7. Cell contents after stirring with a rod. FIG. 8. After a different stirring.

With the observation cell surrounded by a water jacket the temperature gradients could be eliminated. Various patterns could then be produced by jarring the cell slightly. Yet whatever patterns were formed, all tended to persist long after the jarring had ceased. However, the patterns eventually grew less sharp, changed form, and almost disappeared, leaving the general appearance of the cell not completely dark but nearly uniform. For instance in a typical case the pattern formed after the cell had been slightly shaken from side to side, not stirred internally, could be followed for over an hour.

It is then certain that double refraction occurs in this case because the particles have been suitably oriented, and that having been so arranged they tend to maintain the configuration though ultimately the ordinary molecular disturbances will again result in bringing about a random orientation.

In both cases the shearing stresses involved under the conditions are very small, far below values required to produce the double refraction of the type discussed by Maxwell. Hence the conclusion that the double refraction observed by Kundt is of two kinds.

The patterns observed in the cells are in themselves of no particular importance, because it is impossible to deduce therefrom anything definite as to the orientation of particles in a particular region, the phase difference being an integrated value along the light path through a heterogeneous medium. However, it is possible that the results may be of interest in studying some aspects of disturbances in fluids, and several photographs of observed patterns are appended. For these photographs I am indebted to Dr. G. N. Glasoe, who has succeeded in making satisfactory exposures, although both the colloids used, vanadium pentoxide and benzopurpurin, transmit only narrow bands in the red.

My thanks are also due to Professor A. W. Thomas of the chemistry department for the preparation of the colloidal solutions, and for discussions of the problem.

On the Creep of Solids at Elevated Temperature*

A. NADAI
Westinghouse Electric and Manufacturing Company, East Pittsburgh, Pa.
(Received March 26, 1937)

In this paper equations are set up which describe the behavior of a solid under the action of combined stresses when a steady state of creep is assumed to exist. A material is said to be in a steady state of creep if it yields with a constant speed under a given constant stress. In an ordinary creep test the steady state exists when the deformation has reached the point where the "creep curve" becomes a straight line. In mathematical language this can be expressed by saying that the stress is a function of the rate of strain and is independent of the amount of strain. Cases are worked out in which this stress *vs.* strain rate relation is a simple power function. Various types of flow including purely plastic flow and viscous flow can be discussed by changing the value of the exponent in the power function relation. Several specific examples which include the case of the thick walled cylinder with closed ends loaded by internal pressure and the case of a disk having a circular hole and loaded in its plane are worked out. Equations are developed which show the steady state stresses in such cases. Expressions for the stress concentration factors in creep are also obtained.

INTRODUCTION

ON account of the great variety in the mechanical behavior of solids and of the metals as engineering materials in particular the impossibility of ascribing to them simple general mechanical ideal properties such as can be attributed in fluid mechanics to liquids has been recognized. It seems that progress has rather been made in the opposite direction, namely by confining the treatment to smaller groups of solids, by limiting the ranges of the variables, such as the strains, the strain velocities and the temperatures and by describing the plastic deformations independently from the strength properties (for example the breaking conditions, the fatigue phenomena, etc., need separate treatments).

It was mentioned elsewhere that the treatment of the plastic deformations of solids may be restrained either to cases of plastic flow in which the yield stresses remain independent of the speed of deformation or to cases in which they do not depend on the strains, but only on the strain rates, i.e., on the speeds with which the strains change with time. While the former kind of flow is encountered in the polycrystalline ductile metals at comparatively low temperatures in the strain-hardening range, the latter flow is characteristic of the slow creep of metals at elevated temperatures in the creep range. This latter group of the permanent deformations of solids has been experimentally studied by engineers during the past years intensively for the ductile metals at elevated temperatures in connection with creep. While the present paper primarily was concerned with this latter group of phenomena and with the creep of metals under combined stresses, it was later found that the formulae which were derived for the steady state of distributions of stress assuming certain speed relations valid for creep would express also the distributions of stress in the strain-hardening range of metals under the same loading conditions for a corresponding strain-hardening law. This will be explained later in more detail.

Further restrictions refer to the order of magnitude of the permanent strains (whether infinitesimal or finite) and to the type of strains, which have to be considered in certain cases (whether elastic or permanent or both). In view of the extremely small deformations occurring during creep in the following only infinitesimal strains and strain rates need to be considered. Comparatively little is known even under these simplifying assumptions about the redistribution of the stresses around holes or notches or in heavily stressed, thick walled cylinders or rotating disks when the elastic distribution of stress due to the exposure to high temperatures and to creep is gradually changing with time.[1]

* Presented at the Symposium on Metals jointly sponsored by the American Institute of Physics and the Massachusetts Institute of Technology, Cambridge, January 29, 1937.

[1] A considerable amount of valuable information on the creep of metals of an experimental nature became available during the recent years, which cannot be quoted here in detail. Among the papers which dealt recently with problems of a mechanical nature in creep, the following should particularly be mentioned: R. W. Bailey, "The Utilization of Creep Data in Engineering Design." Insti-

Existence of a steady state of creep was assumed. After a certain time has elapsed, it is assumed that the creep rates do not change with time more under constant stress. The formulae permit to compute perhaps for the first time a few exact expressions for factors of stress concentration in simple cases corresponding to the special creep speed law or strain-hardening curves which were used.

The elastic portions of the strains were neglected. It is expected that by this assumption, no serious error in the values of the stresses in the steady state of creep resulted. It should be expressly noted, that the formulae do not converge in the neighborhood of zero stress to the right expressions.[2] For the purpose of finding if possible simple expressions for some distributions of stress in the steady state of creep, use of certain mean quantities dependent on the six components of stress and of strain and of the strain rates is made. They are the measures of the intensities of stress, plastic distorsion and rate of distorsion.

One of the simplest and at the same time most symmetrical expressions which offers itself for this purpose is the sum of the squares of the three principal stress differences. In the homogeneous stress field in the neighborhood of a point, there are eight symmetrically situated plane sections or four sets of parallel planes in which the shearing stress is equal to

$$\tau_n = \tfrac{1}{3}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]^{\frac{1}{2}} \quad (1)$$

and in which at the same time the normal stress σ_n becomes equal to the mean stress σ

$$\sigma_n = (\sigma_1+\sigma_2+\sigma_3)/3 = \sigma, \quad (2)$$

where σ_1, σ_2, σ_3 designate the principal stresses.

These planes are the octahedral planes with reference to the sides of a cube oriented with its faces normally to the principal stress directions. If one selects a small regular octahedron as an element of the plastic material oriented with its axes parallel to the principal stress directions, the normal pressure σ_n on each of the eight faces of this element is equal to the mean pressure σ. Since this latter has no influence on the yielding, it seems reasonable to consider the other stress component, namely, τ_n as the measure of the stress intensity producing plastic flow or yielding. This stress τ_n defined by Eq. (1) is called the octahedral shearing stress.

Similarly, the unit shear in the octahedral plane

$$\gamma_n = \tfrac{2}{3}[(\epsilon_1-\epsilon_2)^2+(\epsilon_2-\epsilon_3)^2+(\epsilon_3-\epsilon_1)^2]^{\frac{1}{2}}, \quad (3)$$

where ϵ_1, ϵ_2, ϵ_3 are the principal strains is called the octahedral shear. Among the various planes in the neighborhood of a point in which the unit shear is equal to γ_n the eight octahedral planes have the additional remarkable property that the strain in the normal direction to them is

$$\epsilon_n = (\epsilon_1+\epsilon_2+\epsilon_3)/3 \quad (4)$$

and consequently equal to the mean strain. Since for small strains the sum $\epsilon_1+\epsilon_2+\epsilon_3$ represents the volume dilatation and during a plastic deformation the volume does not change appreciably, so that this sum remains equal to zero, we see that

$$\epsilon_n = 0.$$

Therefore, the distorsion of the octahedron consists only in the unit shears γ_n and the intensity of a small plastic deformation is measured by the octahedral shear γ_n.

When a ductile metal is deformed at low temperatures, stress is dependent on the strains and must be increased for maintaining flow. This is expressed by assuming that

$$\tau_n = f(\gamma_n) \quad (5)$$

is a monotonously increasing function. (5) expresses in the case of the metals what is known as "cold working" or "strain-hardening." In the statement (5) is implicitly contained the other

tution of Mech. Eng., London (1935). Cf. also Engineering **140**, 595 (1935). F. K. G. Odqvist, "Theory of Creep Under Action of Combined Stresses with Applications to High Temperature Machinery." Proc. Royal Swedish Institute Eng. Research, No. 141, Stockholm (1936). (A part of a paper communicated also to the Fourth Congress for Applied Mechanics, Cambridge, England, 1934. Cf. Proc. p. 228.) C. R. Soderberg, "The Interpretation of Creep Tests for Machine Design." Trans. A. S. M. E. R. P. 58–15 (1936). Also, "Working Stresses," same journal (1933). Paper APM 55–16 p. 131, and Schweizernche Bauzeitung **104**, 127 (1934). G. M. MacCullough, "Applications of Creep Tests" Trans. A. S. M. E. **55**, 87 (1933). H. Hencky, "The New Theory of Plasticity, Strain Hardening and Creep," Trans. A. S. M. E. **55**, 151 (1933).

[2] For example the differential quotient $\partial\sigma/\partial\epsilon$ when $\sigma=0$, $\epsilon=0$ should become equal to the modulus of elasticity E under a pure tensile stress σ independently of the strain rate. The assumptions which were introduced disregard this fact.

observation that the speed with which the plastic deformations were produced was not essential for the magnitude of the stress. In the special case of (5) when a substance yields under a constant octahedral shearing stress, or when

$$\tau_n = \sqrt{2}\sigma_0/3 = \text{const.} \tag{6}$$

the substance is called *perfectly plastic*. The normal stress σ_0 is the plastic limit in pure tension. σ_0 is the normal stress under which the substance would yield under tension.

At elevated temperatures, on the contrary, stress is more dependent on the speed of deformation, less on the strains and one may assume as a second fundamental form of flow *a speed dependent plastic flow* defined by a relation

$$\tau_n = f(\dot{\gamma}_n), \tag{7}$$

where $\dot{\gamma}_n = d\gamma_n/dt$ designates the rate of change of the octahedral infinitesimal shears $\gamma_n \cdot \dot{\gamma}_n$ is the octahedral rate of shear.

When the stress τ_n is increasing in the same proportion as the shear rates $\dot{\gamma}_n$ are increasing, the substance is called *perfectly viscous:*

$$\tau_n = \eta \cdot \dot{\gamma}_n, \tag{8}$$

where η is a material constant.

It is admitted that in the preceding Eqs. (5)–(8) and in the applications to follow, emphasis is laid on ideal conditions of flow. Eq. (7) and some of its consequences will particularly be investigated in the hope of finding simple distributions of stress in the steady state of creep for a few cases of practical interest. This is done also in the hope of verifying some of the formulae derived after assuming a more general creep speedlaw through the two limiting special cases of flow, which have already been quoted referring to Eqs. (6) and (8), respectively. Inspection of Eqs. (5) and (7) expressing conditions of flow has shown that these two more general types of flow must include as special cases the perfectly plastic substance characterized by Eq. (6) and the perfectly viscous substance characterized by Eq. (8). If solutions based on the two more general Eqs. (5) and (7) are later established, they must contain as special cases the solutions which would have been obtained if they would have been based on the conditions (6) or (8), respectively.

It will become apparent that further connections exist between the two Eqs. (5) and (7) defining strain and speed dependent plastic flow respectively and between the well-established laws for a perfectly elastic substance on one side with the law for viscous flow on the other side and finally with the law for perfectly plastic flow. For such a comparison it is necessary for quoting here briefly some partially well-known experimental facts.

The Stress Strain and the Stress-Strain Rate Relations for Plastic Flow

When a material element is permanently deformed, the rates of shear $\partial\gamma/\partial t$ at any instant are proportional to the corresponding shearing stresses τ, independently of the orientation of the planes of shear or of slip. This is expressed by

$$\partial\gamma/\partial t = \tau/\eta, \tag{9}$$

valid for any direction of the planes of shear.

The corresponding is true for the unit shears γ in an elastic material:

$$\gamma = \tau/G \tag{10}$$

with the difference that the proportionality factor G (modulus of rigidity) is a material constant. Applying (10) to the case of pure tension in an elastic substance, for example when a normal stress σ_x acts parallel to the axis and the unit strains ϵ_x in the same direction and the strains ϵ_y and ϵ_z in two perpendicular directions y and z are

$$\epsilon_x = \sigma_x/E, \quad \epsilon_y = \epsilon_z = -\nu\epsilon_x \tag{11}$$

(E modulus of elasticity, ν Poisson's ratio), since (10) must hold also for the maximum shear $\gamma_{\max} = \epsilon_x - \epsilon_y$ and the corresponding shearing stress is $\tau_{\max} = \sigma_x/2$

$$\epsilon_x - \epsilon_y = (1+\nu)\sigma_x/E = \tau_{\max}/G = \sigma_x/2G$$

the relation for the elastic constants follow:

$$E = 2(1+\nu)G. \tag{12}$$

Consequently three normal stresses, σ_x, σ_y, σ_z acting in an elastic material in the directions of x, y, z produce in the x direction the strain

$$\epsilon_x = (\sigma_x - \nu\sigma_y - \nu\sigma_z)/E \tag{13}$$

and adding the three similar expressions for ϵ_x, ϵ_y, ϵ_z gives

$$e=3(1-2\nu)\sigma/E, \tag{14}$$

where e stands for the elastic volume dilatation $e=\epsilon_x+\epsilon_y+\epsilon_z$ and σ is given by (1). When $\nu=\frac{1}{2}$, $e=0$ and

$$E=3G. \tag{15}$$

Similarly, for establishing the stress rate of strain relations valid for a steady plastic flow or for creep, we note that when a normal stress σ_x acts, the material flows with a rate of strain $u_x=\partial\epsilon_x/\partial t$ in the direction of the x axis and contracts in the perpendicular directions with the strain rates

$$u_y=\partial\epsilon_y/\partial t \quad \text{and} \quad u_z=\partial\epsilon_z/\partial t.$$

Since for small permanent strains $e=\epsilon_x+\epsilon_y+\epsilon_z=0$ also the sum of the three permanent strain rates

$$u_x+u_y+u_z=0 \tag{16}$$

must vanish. Therefore, a normal stress σ_x must produce the following strain rates:

$$u_x=\sigma_x/\phi, \quad u_y=u_z=-u_x/2=-\sigma_x/2\phi. \tag{17}$$

Since the maximum rate of shear corresponding to this simple distorsion is $u_x-u_y=3u_x/2$ and $\tau_{max}=\sigma_x/2=\phi u_x/2$ on account of Eq. (9): $\tau_{max}=\eta(u_x-u_y)$ and we see that for the permanent strain rates:

$$\phi=3\eta. \tag{18}$$

Consequently the plastic stress-strain rate relations when the normal stresses σ_x, σ_y and σ_z act simultaneously, must be

$$\left.\begin{aligned} u_x&=[\sigma_x-(\sigma_y+\sigma_z)/2]/\phi,\\ u_y&=[\sigma_y-(\sigma_z+\sigma_x)/2]/\phi,\\ u_z&=[\sigma_z-(\sigma_x+\sigma_y)/2]/\phi. \end{aligned}\right\} \tag{19}$$

To these come the relations for the shearing stress components expressed by Eq. (9). We will designate in the following shear rates with the letter v, so that v stands for $\partial\gamma/\partial t$. For the shearing stress therefore

$$v=\tau/\eta. \tag{20}$$

We note that the Eqs. (19) satisfy already the condition of incompressibility Eq. (16), so that (16) needs not be introduced more, and that the quantities ϕ and η which have been introduced in the plastic stress-strain rate relations (19) and (20) are analogous to the familiar moduli of elasticity and rigidity E and G, respectively, appearing in the stress-strain relations (10) and (13) of an elastic substance. Since both substances are assumed as incompressible we have

$$\phi=3\eta \quad \text{and} \quad E=3G.$$

ϕ and η could be called the plastic normal and shearing moduli, but for other reasons becoming soon apparent they have received other designations.

The permanent strains ϵ_x, $\cdots$ and strain rates u_x, $\cdots$ to be considered in this article remain small. Products of these latter quantities where they would be necessary in dynamic equations can be neglected. The forces producing flow or creep can also be assumed to be in equilibrium.

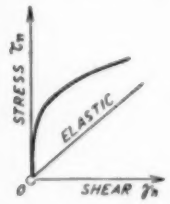
Under these circumstances the correspondence between the group of equations valid for a perfectly elastic substance and that for the steady flow of a perfectly viscous substance ϕ, $\eta=$const becomes apparent. All that is necessary for establishing full correspondence is to assume incompressibility for the elastic material. The following variables and material constants correspond to each other:

Elastic substance: Strains: ϵ_x, ϵ_y, ϵ_z, γ; Constants: $E=3G$.
Viscous substance: Strain rates: u_x, u_y, u_z, v;
Constants: $\phi=3\eta$.

The quantities ϕ and η in the case of a perfectly viscous substance are known as the coefficients of viscosity for tension (or compression) and for shear, respectively. During the slow steady flow of a perfectly viscous solid substance the distribution of the stresses is described by the same functions of the coordinates x, y, z as that which would be obtained in an incompressible elastic solid under the same boundary conditions.[3] In many cases this correspondence exists even though an ordinary (compressible) elastic solid is substituted for the incompressible one, namely in all those numerous cases of stress more, in which the functions representing the stresses are independent of Poisson's ratio ν. For example, the equation of the elastic line of a beam in bending represents the shape of the bent line of a beam of a viscous material subjected to the same forcefield in the steady state. Or the formulae

[3] See also, T. N. Goodier, "Slow Viscous Flow and Elastic Deformation," Phil. Mag. **22**, 678 (1936).

for the radial and tangential stress σ_r and σ_t in function of the radius r in an elastic cylinder subjected to internal pressure express also the distribution of σ_r and σ_t in a similar cylinder of a



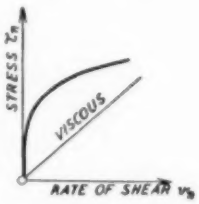


FIG. 1 (left). Strain-hardening curve. FIG. 2 (right). Creep speed law.

viscous material in the steady state. In the formulae for an elastic disk of radius "a" revolving with an angular speed ω

$$\sigma_r=(3+\nu)\gamma_0\omega^2(a^2-r^2)/8g,$$
$$\sigma_t=[(3+\nu)a^2-(1+3\nu)r^2]\gamma_0\omega^2/8g.^4$$

However, ν would have to be taken equal to $\frac{1}{2}$ for obtaining the corresponding stress formula for a revolving disk of a viscous material:

$$\sigma_r=7\gamma_0\omega^2(a^2-r^2)/16g,$$
$$\sigma_t=(7a^2-5r^2)\gamma_0\omega^2/16g.$$

From the stress-strain rate relations for speed dependent flow given by Eqs. (19) and (20) the stress-strain relations for strain dependent flow are obtained after substituting in (19) and (20) for the letters u and v the letters ϵ and γ:

$$\left.\begin{aligned}\epsilon_x&=[\sigma_x-(\sigma_y+\sigma_z)/2]/\phi,\\ \epsilon_y&=[\sigma_y-(\sigma_z+\sigma_x)/2]/\phi,\\ \epsilon_z&=[\sigma_z-(\sigma_x+\sigma_y)/2]/\phi,\\ \gamma&=\tau/\eta.\end{aligned}\right\}\qquad(21)$$

When the strains, respectively, the strain rates are infinitesimal and equilibrium prevails, a strain dependent plastic distortion bears the same relation to the distorsion of an incompressible elastic material than a slow steady speed dependent flow bears to the steady flow of a viscous substance. This is perhaps illustrated bv comparing the graph of the strain-hardening curve (Eq. 5) indicated in Fig. 1 with the graph of the creep speed law (Eq. 7) shown in Fig. 2. Let us assume that both of these curves would correspond to the same mathematical law, i.e., $\tau=f_1(\gamma)$ would be the same function of γ as $\tau=f_2(v)$ is of v. Then the solution expressing the stresses in function of the coordinates is the same for a plastic strain-hardening material as it is in the steady state of creep. If, for example, the strain-hardening law is expressed by

$$\tau=\tau_1(\gamma/\gamma_1)^m$$

and the creep speed law by a power function

$$\tau=\tau_2(v/v_2)^m$$

after a proper selection of the constants and using the same exponent in both functions, the stresses under given boundary conditions would be expressed by the same functions of the coordinates. When for the exponent m is taken $=1$ the curves in Figs. 1 and 2 are straight lines, corresponding to the special cases of the elastic, incompressible and the perfectly viscous substance respectively.

TWO-DIMENSIONAL CREEP FLOW

For the permanent deformations the following equations are available: The strain rates u_x, u_y, u_z and the rate of shear v expressed in terms of the components w_x and w_y of the velocity vector w of a point (x, y) in the solid are

$$u_x=\frac{\partial w_x}{\partial x},\quad u_y=\frac{\partial w_y}{\partial y},\quad v=\frac{\partial w_x}{\partial y}+\frac{\partial w_y}{\partial x}\qquad(22)$$

the condition of compatibility of these three equations is

$$\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}=\frac{\partial^2 v}{\partial x\partial y},\qquad(23)$$

the equilibrium conditions are

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau}{\partial y}=0,\quad \frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau}{\partial x}=0,\qquad(24)$$

the plastic stress-strain rate relations are

$$\left.\begin{aligned}u_x&=(2\sigma_x-\sigma_y-\sigma_z)/6\eta,\\ u_y&=(2\sigma_y-\sigma_z-\sigma_x)/6\eta,\\ u_z&=(2\sigma_z-\sigma_x-\sigma_y)/6\eta,\\ v&=\tau/\eta.\end{aligned}\right\}\qquad(25)$$

We noted already that this last group of relations takes care of the condition of incompressibility expressed by Eq. (16). Similarly as in the theory

[4] γ_0 is the specific weight.

of elasticity a few special cases are of importance in which one additional condition is imposed on the cinematic relations (25) which shall be mentioned now:

a. Plane flow

The normal strain rate u_z in the direction of the z axis is constant. The simplest case is $u_z=0$. From Eq. (16) follows then:

$$u_x=-u_y \tag{26}$$

and from the third of the Eq. (25) that in this case the axial stress σ_z must become equal

$$\sigma_z=(\sigma_x+\sigma_y)/2. \tag{27}$$

Instead of the group of Eqs. (25) the stress-strain rate equations are condensed to:

$$u_x=-u_y=(\sigma_x-\sigma_y)/4\eta, \quad v=\tau/\eta. \tag{28}$$

b. Thin plate or disk $\sigma_z=0$

After inserting this value for σ_z in (25), the following is obtained:

$$u_x=(2\sigma_x-\sigma_y)/6\eta, \; u_y=(2\sigma_y-\sigma_x)/6\eta, \; v=\tau/\eta. \tag{29}$$

If we count now the variables appearing in the Eqs. (23, 24, and 28 or 29, respectively), we find seven: σ_x, σ_y, τ, η, u_x, u_y, v, for which only six equations are available. The one additional equation needed is derived from what was called the "speed" or "creep law" or the material stating the physical relation which must exist between the speed of deformation (the creep rates) and the stresses under which they are produced.

c. Perfectly viscous substance

One special case has already been quoted, namely *that of a perfectly viscous substance:* The viscosity coefficient η in the cinematic relations is constant. To Eq. (22) or $\tau=\eta v$ must be attributed two simultaneous meanings: it expresses the cinematic relation of flow and also the speed law in this particular case known as the law of viscous flow.

d. Perfectly plastic substance. Speed independent flow

In this case η is considered as an unknown function of the coordinates x, and y. One additional equation is still needed which is obtained by assuming a pure stress condition of yielding. For this latter the condition is taken that the shearing stress τ_n in the octahedral planes is equal to a constant or that

$$(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2=9\tau_n^2=2\sigma_0^2. \tag{30}$$

As experiments have shown it represents fairly well the yielding of the ductile metals at comparatively low temperatures. (σ_1, σ_2, σ_3 are the principal stresses, σ_0 is the yield stress in pure tension.)

In the case of two dimensional flow of a perfectly plastic substance stresses σ_x, σ_y, τ can now directly be computed from (24) and (30) and the remaining equations serve for the evaluation of u_x, u_y, v, η. The compatibility equation shows, however, that in this case the $u_x u_y v$ can be multiplied with an arbitrary factor. These quantities thus remain undertermined. With other words the velocities with which the substance flows can be increased or diminished in the same proportion in all points, without altering the values of stresses during yielding.

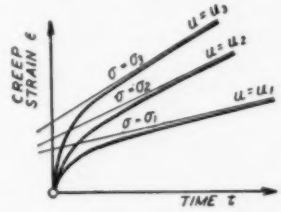


FIG. 3. Steady states of creep.

e. Steady creep obeying given law. Speed dependent flow

In the steady stage of creep the existence of a function

$$\tau_n=f(v_n) \tag{31}$$

was assumed in which τ_n and v_n are the shearing stress and the rate of shear in the octahedral planes which were defined by the expressions

$$\tau_n^2=(1/9)[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2], \tag{32}$$

$$v_n^2=(4/9)[(u_1-u_2)^2+(u_2-u_3)^2+(u_3-u_1)^2]. \tag{33}$$

Eq. (31) is based on the fact that the creep time curves $\epsilon=f_1(t)$ derived from observation in tensile tests under a constant stress $\sigma=$const. seem to flatten out (Fig. 3) so that after sufficient time t has elapsed they might in the first approximation be considered to consist of a family of inclined straight lines $\epsilon=a+bt$, the slopes $u=\partial\epsilon/\partial t$ of

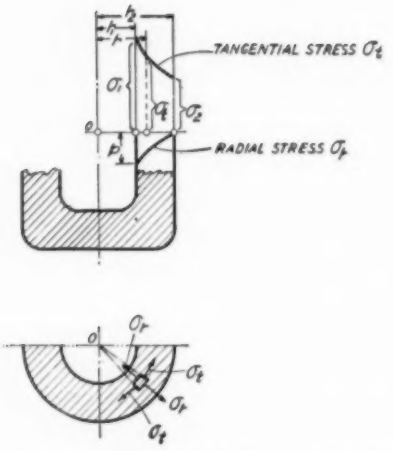


FIG. 4 (top). FIG. 5 (bottom). Stresses in thick walled cylinders under internal pressure.

which when plotted against stress furnish a relation $\sigma = f_2(u)$ similar to Eq. (31). This therefore refers only to "the steady state of creep" and the following theory is not capable of predicting the conditions in the earlier transition period.

ROTATIONAL SYMMETRICAL CREEP

a. Creep in cylinder with closed ends under internal pressure

We shall consider first the case that in axial direction creep does not occur, i.e., that the strain rate in the direction z parallel to the axis of rotation $u_z = 0$. Introducing the radius r as coordinate, σ_r and σ_t for the radial and tangential stress (Figs. 4 and 5), u_r and u_t for the radial and tangential strain rate, the cinematic relations according to Eq. (28) are

$$u_t = -u_r = (\sigma_t - \sigma_r)/4\eta, \tag{34}$$

the compatibility condition is

$$(d/dr)(ru_t) - u_r = 0, \tag{35}$$

and the equilibrium condition

$$(d/dr)(r\sigma_r) - \sigma_t = 0. \tag{36}$$

To these four equations for the five unknowns $u_r u_t \sigma_r \sigma_t \eta$ must now be added a statement about the speed law. We have first to compute the octahedral shearing stress τ_n and the shear rate v_n. Omitting the subscript when these latter quantities are expressed and noting that σ_r, σ_t, σ_z are principal stresses, u_r, u_t, $(u_z = 0)$ principal strain rates and finally that on account of $u_z = 0$, $u_r = -u_t$, and $\sigma_z = (\sigma_x + \sigma_y)/2$ the octahedral shearing stress from Eq. (32) is found equal to

$$\tau = (\sigma_t - \sigma_r)/\sqrt{6} \tag{37}$$

and the octahedral shear v from Eq. (33) is found equal to

$$v = 2[2(u_r^2 + u_t u_r + u_t^2)/3]^{\frac{1}{2}} = 2(2/3)^{\frac{1}{2}} \cdot u_t. \tag{38}$$

For the creep law

$$\tau = \tau^*(v/v^*)^m \tag{39}$$

is introduced where τ^*, v^* and m are constants. If $\tau_1 v_1$, $\tau_2 v_2$, $\cdots$ are corresponding values, then (39) can also be written as

$$\tau = \tau_1(v/v_1)^m = \tau_2(v/v_2)^m. \tag{40}$$

Either of the pairs of constants: τ^*, v^*, $\tau_1 v_1$, $\tau_2 v_2$, $\cdots$ can be considered as the "material constants" of the creep law. The exponent is $0 \leqq m \leqq 1$. We note that when (37) is divided by (38) and use is made of (34) between the octahedral shearing stress τ and shear v the second equation

$$\tau = \eta v \tag{41}$$

must be obtained because the cinematic relations must also refer to the octahedral planes of shear in the stress tensor. Combining now the two expressions (40) and (41) η can be expressed in terms of τ alone

$$\eta = \eta_1(\tau_1/\tau)^{1/(m-1)}. \tag{42}$$

Here τ is replaced using (37) by the stress difference $\sigma_t - \sigma_r$

$$\eta = \eta_1\left[\frac{(\sqrt{6}) \cdot \tau_1}{\sigma_t - \sigma_r}\right]^n, \tag{43}$$

η_1 designates the value of $\eta_1 = \tau_1/v_1$, and the new exponent n introduced in (43) is

$$n = 1/m - 1. \tag{44}$$

When m varies between 0 and 1 the new exponent n varies between ∞ and 0.

On account of $u_t = -u_r$ the compatibility Eq. (35) can immediately be integrated giving

$$u_t = (\sigma_t - \sigma_r)/4\eta = c/r^2, \tag{45}$$

the integration constant c is found from the boundary condition

$$r = r_1, \quad \sigma_r = -p, \quad \sigma_t = \sigma_1, \tag{46}$$

which gives $$c = (\sigma_1 + p)r_1^2/4\eta_1. \tag{47}$$

Computing the value of τ_1 from (37)

$$\tau_1=(\sigma_1+p)/\sqrt{6}$$

and inserting τ_1 in (43) and c from (47) in (45) we obtain

$$\frac{\eta}{\eta_1}=\left(\frac{\sigma_1+p}{\sigma_t-\sigma_r}\right)^n \tag{48}$$

and

$$\frac{\eta_1}{\eta}=\frac{(\sigma_1+p)r_1^2}{(\sigma_t-\sigma_r)r^2}. \tag{49}$$

These last two expressions allow to compute the stress difference $\sigma_t-\sigma_r$ and the viscosity coefficient in terms of r:

$$\sigma_t-\sigma_r=(\sigma_1+p)(r/r_1)^{-2/(n+1)}, \tag{50}$$

$$\eta=\eta_1\cdot(r/r_1)^{2n/(n+1)}. \tag{51}$$

After using the value for the stress difference $\sigma_t-\sigma_r$ which has just been computed, the equilibrium condition (36) can be integrated and furnishes with the second boundary condition $r=r_2$, $\sigma_r=0$ the unknown radial stress σ_r. With the abbreviations

$$\rho=r/r_2, \quad \rho_1=r_1/r_2 \tag{52}$$

the following exact expressions for the steady radial and tangential creep stresses in a heavy walled cylinder subjected to an internal pressure p are obtained

$$\left.\begin{aligned}\sigma_r&=-c[\rho^{-2m}-1],\\ \sigma_t&=c[-(1-2m)\rho^{-2m}+1],\\ \text{where}\quad c&=p\rho_1{}^{2m}/(1-\rho_1{}^{2m}).\end{aligned}\right\} \tag{53}$$

The tangential stresses σ_t at the inner and at the outer surface of the cylinder are found equal to:

$$\left.\begin{aligned}\rho=\rho_1\cdots\sigma_1&=(2m-1+\rho_1{}^{2m})p/(1-\rho_1{}^{2m}),\\ \rho=1\cdots\sigma_2&=2m\rho_1{}^{2m}p/(1-\rho_1{}^{2m}).\end{aligned}\right\} \tag{54}$$

Some of these stress distributions are plotted in Fig. 6 assuming for the exponents m the values: $m=1$, $\frac{2}{3}$, $\frac{1}{2}$, $\frac{1}{4}$ and 0. The corresponding curves representing the creep speed laws are given in Fig. 7. The stress formulae (53) when m is chosen equal to 1 are:

$$\sigma_r=-\frac{p\rho_1{}^2(\rho^{-2}-1)}{1-\rho_1{}^2}, \quad \sigma_t=\frac{p\rho_1{}^2(\rho^{-2}+1)}{1-\rho_1{}^2}. \tag{55}$$

These coincide with the well-known expressions for the stresses *in a viscous or an elastic tube*. This was to be expected since when m is taken equal to 1 according to the speed or creep law (39)

$$\tau=\tau_1 v/v_1$$

stress τ is proportional to the shear rate v and the case when $m=1$ evidently corresponds to flow of a tube of a perfectly viscous substance.

When on the other hand the exponent m is approaching the value zero the diagram representing the creep laws Fig. 7 shows that τ tends to become a constant $\tau=\tau_1$, the shearing stress τ is

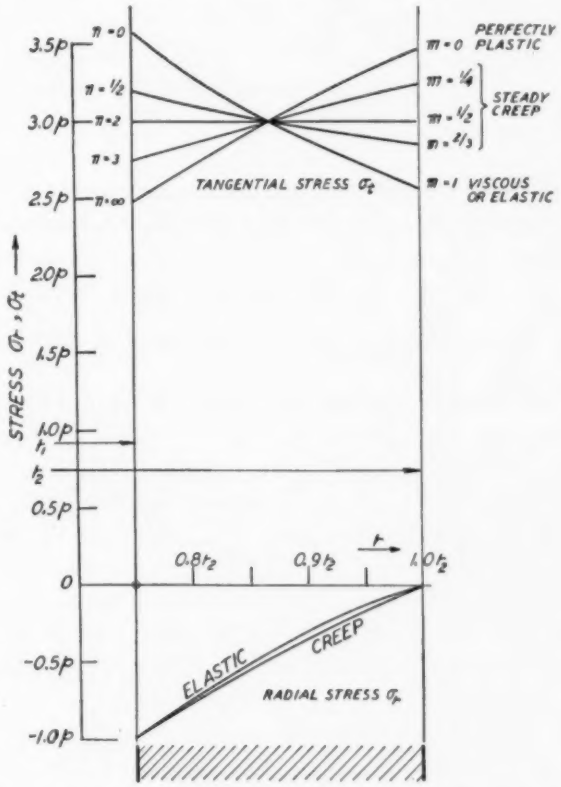


FIG. 6. Steady stress distribution in heavy walled cylinder after creep.

becoming independent of the shear rate $v\cdot m=0$ therefore corresponds to *flow in a tube of a perfectly plastic substance*.

When $m=0$ the stresses appear in the form $0/0$ but their expressions can easily be derived from (36) by direct integration:

$$\sigma_r=-p\frac{\ln\rho}{\ln\rho_1}, \quad \sigma_t=-p\cdot\frac{(1+\ln\rho)}{\ln\rho_1}. \tag{56}$$

We see furthermore that an exponent m exists for

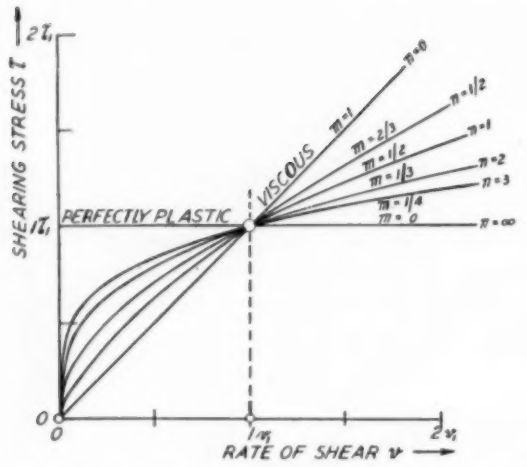


FIG. 7. Shearing stress τ *versus* rate of shear v for various speed laws $\tau \cdot \tau_1(v/v_1)^m$; $m=0$ perfectly plastic substance; $m=1$ perfectly viscous substance.

which the steady tangential creep stresses in the tube wall are uniformly distributed. This occurs when

$$m=\tfrac{1}{2}, \quad \sigma_t = p\rho_1/(1-\rho_1) = \text{const.} \qquad (57)$$

(σ_1 in this case can even become smaller than p, when $\rho_1 < \frac{1}{2}$).

These three special cases of the formulae (53) have also been plotted in Fig. 4, which gives an idea of the σ_t curves to be expected in a heavy walled cylinder for the various creep laws when the steady state has been attained. The figure discloses an interesting result, namely that the maximum stress $\sigma_{t\,\max}$ which in an elastic tube is found at the inner surface after creep took place may shift to the outer surface of the tube. The stress σ_1 at the inner surface becomes larger than the stress σ_2 at the outer surface as long as the exponent in the creep law (39) is $\frac{1}{2} < m < 1$. Fig. 6 for the stresses in a tube contains the two extreme cases of flow for a perfectly viscous ($m=1$) and for a perfectly plastic ($m=0$) substance.

With regard to the axial stress σ_z one remark is still in order. We treated the case making the assumption that the tube yields or flows without changing lengths in the axial direction, i.e., when $u_z = \epsilon_z = 0$. In this case the axial stress according to Eq. (27) is $\sigma_z = (\sigma_r + \sigma_t)/2$.

Using (53) this gives

$$\sigma_z = -\frac{pm\rho_1^{2m}}{(1-\rho_1^{2m})\rho^{2m}}. \qquad (58)$$

If now using this σ_z the mean value of the axial stress $\bar{\sigma}_z$ over the tube wall or

$$\bar{\sigma}_z = 2\int_{\rho_1}^{1} \sigma_z \rho d\rho/(1-\rho_1^2) \qquad (59)$$

is computed, it turns out to be independent of the exponent m, namely equal to:

$$\bar{\sigma}_z = p\rho_1^2/(1-\rho_1^2). \qquad (60)$$

A tube with closed ends subjected to internal pressure p has the same mean axial stress $\bar{\sigma}_z = pr_1^2/(r_2^2-r_1^2) = p\rho_1^2/(1-\rho_1^2)$. We must conclude therefore that our condition $u_z=0$ imposed at the beginning on the stress distribution expressed by the Eqs. (53) is identical with the other condition that the mean axial stress $\bar{\sigma}_z$ in the tube is so large that it just balances the axial tensions which would be produced when closing both ends of the tube. It can indeed easily be shown that this is not only true for the *resultant* of the axial stresses σ_z but that the condition $u_z=0$ is satisfied in any tube with closed ends subjected to internal pressure p. Indeed when

$$\sigma_t = (d/dr)(r\sigma_r) \quad \text{and when} \quad \sigma_z = (\sigma_r+\sigma_t)/2$$

the integral

$$\int \sigma_z r dr = \tfrac{1}{2}\int \sigma_r r dr + \tfrac{1}{2}\int [d(r\sigma_r)/dr]\cdot r dr.$$

Using partial integration and $r=r_1$ and $r=r_2$ as limits this is found equal to

$$\int \sigma_z r dr = pr_1^2/2.$$

The mean stress $\bar{\sigma}_z$ is therefore always $pr_1^2/(r_2^2-r_1^2)$. This is the same value as obtained in a closed tube. In such a tube therefore the condition $u_z=0$ prevails. According to what was said before, we note finally that the same distribution of stresses in a cylinder as shown in Fig. 6 and as found in the steady state of creep using as speed laws the power function law, Eq. (39), Fig. 2 and expressed by (53) would have been derived in a tube of a plastic material flowing under pressure the strain-hardening curve of which for cold working would be expressed by a power function

$$\tau = \tau^*(\gamma/\gamma^*)^m, \quad 0 < m < 1$$

of the shears γ or a set of curves similar to those shown in Fig. 7.

Stresses around cylindrical cavity

Let r_1 be the radius of the cavity and let an equal stress σ act in larger distance from it in the directions perpendicular to the axis of the cavity. Let $u_z=0$. The boundary conditions to be satisfied by the radial stress are $r=r_1$, $\sigma_r=0$ and $r=\infty$, $\sigma_r=\sigma=$const. The integration constant c in Eq. (45) is found from the first of these conditions. When $r=r_1$ let $\sigma_t=\sigma_1$, $\eta=\eta_1$; then $c=\sigma_1 r_1^2/4\eta_1$. Eqs. (46)–(51) give

$$\sigma_t-\sigma_r=\sigma_1\rho^{-2m}, \quad \eta=\eta_1\rho^{2(1-m)}, \tag{61}$$

where ρ stands now as abbreviation for the ratio

$$\rho=r_1/r.$$

The radial and tangential stress are now given by the formulae:

$$\left.\begin{aligned}\sigma_r&=\sigma[1-\rho^{-2m}],\\ \sigma_t&=\sigma[1+(2m-1)\rho^{-2m}].\end{aligned}\right\} \tag{62}$$

At the surface of the cavity $r=r_1$, $\rho=1$ the tangential stress is

$$\sigma_1=2m\sigma. \tag{63}$$

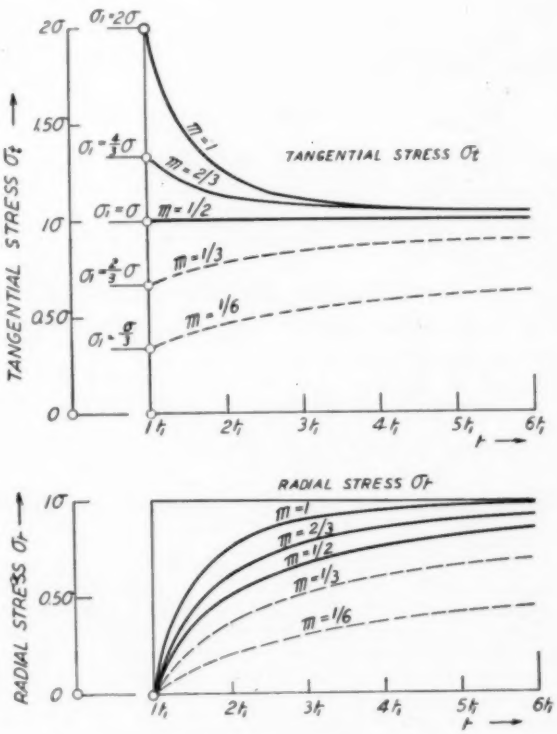


FIG. 8. Steady stresses around cylindrical cavity for various speed laws. No axial rate of flow: $U_z=0$ stress concentration factor $K=2m$.

The stress concentration factor k defined as usually is the ratio:

$$k=\sigma_1/\sigma=2m. \tag{64}$$

k is thus found to be equal to twice the exponent m appearing in the creep speed law:

$$\tau=\tau_1(v/v_1)^m.$$

For a few values of m the steady creep stresses according to (62) are plotted in Fig. 8 around the cylindrical cavity. For m was taken

$$m=1,\ \tfrac{2}{3},\ \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{6}.$$

The value of $m=1$ corresponds again to a perfectly viscous substance, the distribution for σ_r and σ_t when $m=1$ is identical with the one around a cavity in an elastic material. The factor $k=2$.

An interesting case occurs when m is taken equal to $\frac{1}{2}$. The tangential stresses are redistributed in a uniform manner and

$$\sigma_r=-\sigma(1-\rho)/\rho, \quad \sigma_t=\sigma=\text{const.} \tag{65}$$

When the exponent m in the creep speed law is smaller than $\frac{1}{2}$ the stress σ_1 at the cavity becomes smaller than σ_2.[5]

[5] In order to check the formulae (62) further the octahedral shearing stress τ_1 was computed at the bore. τ_1 is found from (37) taking $\sigma_r=0$, $\sigma_t=\sigma_1=2m\sigma$ which gives

$$\tau_1=\sigma_1/\sqrt{6}=(\tfrac{2}{3})^{\frac{1}{2}}\cdot m\sigma. \tag{66}$$

When m tends to approach the value zero, the octahedral shearing stress according to the definition of a perfectly plastic substance Eq. (8) must become equal to $\sqrt{2}\cdot\sigma_0/3$; σ_0 is the yield stress in pure tension. Making $\tau_1=\sqrt{2}\cdot\sigma_0/3$ for σ the value

$$\sigma=\sigma_0/\sqrt{3}\cdot m \tag{67a}$$

and after inserting this in the expression for σ_r (Eq. 62)

$$\sigma_r=\sigma_0(1-\rho^{-2m})/\sqrt{3}\cdot m \tag{67b}$$

is obtained. If the exponent m tends now to become zero, on account of

$$\lim_{m=0}\left(\frac{1-\rho^{-2m}}{m}\right)\overset{0}{=}\overset{}{=}2\ln\rho$$

the formula (67b) for the radial stress σ_r converges into the function:

$$\sigma_r=(2\sigma_0/\sqrt{3})\ln\rho. \tag{68}$$

Similarly when $m=0$ σ_t converges into the function:

$$\sigma_t=(2\sigma_0/\sqrt{3})(\ln\rho+1). \tag{69}$$

These are the known formulae for the stresses around a cylindrical cavity pierced through a perfectly plastic substance. As can be seen from (67a) the stress σ in this last case ($m=0$) increases beyond all limits, so that cases when m tends to approach the value zero are of no practical interest.

Disk Having Circular Hole and Stretched in Its Plane. Stress Concentration Factor in Creep

Let the radius of the hole be r_1 and the stress for large values of r be equal to $\sigma_r=\sigma$. It is required to find σ_r and σ_t as functions of r and the value of the tangential stress σ_1 at the bore after a redistribution of the stresses in the disk took place and the steady state of creep is reached. As usually in such cases a factor of stress concentration $k=\sigma_1/\sigma$ can be defined and it is particularly required to find the value of k for a given exponent m of the creep speed law:

$$\tau=\tau_1(v/v_1)^m. \tag{70}$$

The normal stress in the perpendicular direction to the plane of the disk is zero: $\sigma_z=0$. For the radial and tangential stress and strain rate the following equations are available:

$$u_r=(2\sigma_r-\sigma_t)/6\eta, \tag{71}$$

$$u_t=(2\sigma_t-\sigma_r)/6\eta. \tag{72}$$

The compatibility condition

$$d(ru_t)/dr=u_r, \tag{73}$$

the equilibrium condition

$$d(r\sigma_r)/dr=\sigma_t. \tag{74}$$

The octahedral shearing stress τ defined by Eq. (30), on account of $\sigma_z=0$ is given by the expression

$$\tau=\tfrac{1}{3}\cdot[2(\sigma_t^2-\sigma_t\sigma_r+\sigma_r^2)]^{\frac{1}{2}} \tag{75}$$

and the shear rate v by

$$v=2\cdot[\tfrac{2}{3}(u_t^2+u_tu_r+u_r^2)]^{\frac{1}{2}}. \tag{76}$$

From these 7 equations by gradual elimination two simultaneous equations for the two unknown stresses σ_r and σ_t can be obtained. One more relation already implicitly contained in the stress-strain rate relations, namely

$$\tau=\eta v \tag{77}$$

allows the elimination of v. From (70) and (77):

$$\eta=\eta_1(\tau_1/\tau)^n, \quad (n=1/m-1). \tag{78}$$

General integration of the preceding group of equations becomes possible after making a slight change in the expression of τ in Eq. (75). When $r=r_1$, $\sigma_r=0$ and when $r=\infty$, $\sigma_r=\sigma_t=\sigma$. We observe that at both ends of the interval $r_1<r<\infty$ the right side of Eq. (75) becomes the same expression, so that there

$$\tau=\sqrt{2}\cdot\sigma_t/3. \tag{79}$$

This suggests to make τ in the whole disk equal to this last expression instead of using for τ Eq. (75).[6]

Substituting τ from (79) into (78) gives

$$\eta=\eta_1(\sigma_1/\sigma_t)^n. \tag{80}$$

σ_1 and η_1 are here the values of σ_t and η at the bore $r=r_1$. Eliminating the strain rates from (71), (72) and (73) gives

$$r\frac{d}{dr}\left(\frac{2\sigma_t-\sigma_r}{\eta}\right)+\frac{3}{\eta}(\sigma_t-\sigma_r)=0, \tag{81}$$

and after substituting in this equation the value for η from (80) the first of the following two differential equations for σ_r and σ_t is obtained while the second is Eq. (74)

$$r(d/dr)[\sigma_t^n(2\sigma_t-\sigma_r)]+3\sigma_t^n(\sigma_t-\sigma_r)=0, \tag{82}$$

$$r(d\sigma_r/dr)+\sigma_r-\sigma_t=0. \tag{83}$$

Using instead of r the ratio

$$\rho=r/r_1 \tag{84}$$

and after changing the variables $\sigma_r\sigma_t$ and ρ according to:

$$x=2\sigma_t-\sigma_r, \quad y=\sigma_t-\sigma_r, \quad z=\ln\rho. \tag{85}$$

Eqs. (82) and (83) are transformed in the two differential equations for x and y in function of the new independent variable z:

$$nx(x'-y')+(x'+3y)(x-y)=0, \tag{86}$$

$$x'-2y'-y=0. \tag{87}$$

The radial and the tangential stress σ_r and σ_t are given by

$$\sigma_r=x-2y, \quad \sigma_t=x-y. \tag{88}$$

[6] Examples show that in a thin disk stretched in its plane by tensile stresses the values of τ computed in both ways (Eq. (75) or (79)) do not differ much. For a perfectly plastic material the quantity $\tau=\sqrt{2}\sigma_0/3$ is const. (σ_0 yield stress in tension) and (75) represents an ellipse in the coordinates σ_r, σ_t. The points corresponding to the states of stress which are possible in the disk under the given assumptions are found in a comparatively short arc of the ellipse. According to Eq. (79) this arc is replaced by a secant situated near to the ellipse.

A set of particular integrals of (86) is easily found assuming that

$$y=\alpha x, \tag{89}$$

where α does not contain z. Substituting (89) in (86) and replacing $1+n$ by $1/m$

$$(1-\alpha)x[x'+3m\alpha x]=0, \tag{90}$$

$$(1-2\alpha)x'-\alpha x=0 \tag{91}$$

is obtained. Eq. (90) is satisfied when either $\alpha=1$ or when $x=0$ or when the bracket is equal to zero. When $\alpha=1$,

$$x'+x=0$$

and $$x=y=ce^{-z}=ce^{-\ln\rho}=c/\rho$$

is a set of particular integrals. The corresponding stresses according to (88) are

$$\sigma_r=-c/r, \quad \sigma_t=0.$$

This is the radial distribution of stress in a sector which has been cut out of the disk. The sector is pulled in the radial directions only, the straight sides being free of stress. The second case (when $x=y=0$) is trivial. In the third case however when the bracket in (90) vanishes, we have the two homogeneous linear equations for x' and x:

$$x'+3\alpha m x=0,$$
$$(1-2\alpha)x'-\alpha x=0.$$

With the condition that their determinant vanishes: $\alpha(1+3m-6m\alpha)=0$. When $\alpha=0$, $\sigma_r=\sigma_t=c$, which is again a trivial case. However, when the second factor vanishes, we have $\alpha=(1+3m)/6m$ and $x'=-3m\alpha x$ so that

$$x=y/\alpha=ce^{-3m\alpha z}=c/\rho^{3m\alpha} \tag{92}$$

is a solution. In this case the boundary condition that for $\rho=1$, $\sigma_r=-p$ can be satisfied. The corresponding expressions for the stresses:

$$\sigma_r=-p\cdot\rho^{-(1+3m)/2}, \quad \sigma_t=\frac{(3m-1)}{2}p\rho^{-(1+3m)/2} \tag{93}$$

show that this is the distribution of stresses in a large disk having a hole with the radius r_1. The inside of the hole is subjected to the pressure p in radial direction.[7]

[7] On account of the practical impossibility to apply large pressures in the plane of a thin disk without producing an instability and buckling, this case need not to be dealt with further. Taking $m=1$ the solution is identical with one in an elastic or a viscous material.

Returning to the integration of (86) and (87) we eliminate x' from these two equations, obtaining

$$x=\frac{2y(y'+2y)}{(n+2)y'+(n+4)y}. \tag{94}$$

From (87) we see that also

$$x'=dx/dz=(dx/dy)\cdot y'=2y'+y. \tag{95}$$

After substituting now for

$$y'=y/w. \tag{96}$$

Eqs. (94) and (95) can be rewritten in the form:

$$x=\frac{2y(1+2w)}{n+2+(n+4)w} \tag{97}$$

and $$dx/dy=2+w. \tag{98}$$

If x from (97) is now inserted in (98) a differential equation connecting the variables y and w:

$$\frac{d}{dy}\left[\frac{(1+2w)y}{n+2+(n+4)w}\right]=1-\frac{w}{2} \tag{99}$$

is obtained. Eq. (99) can further be simplified by making use of the new variables ω and v:

$$\omega=1-\frac{\sigma_r}{\sigma_t}=\frac{y}{x-y} \quad \text{and} \quad v=\ln y. \tag{100}$$

From (100)

$$x=\frac{1+\omega}{\omega}\cdot y. \tag{101}$$

w is expressed by means of ω after equating (101) and (97):

$$w=-\frac{n+2+n\omega}{n+4+n\omega} \tag{102}$$

and this enables us to rewrite finally (99) using ω and v as variables:

$$\frac{d\omega}{dv}=\omega\cdot\frac{n+4+(n-2)\omega}{n+4+n\omega}. \tag{103}$$

Integration of (103) gives finally:

$$v=\ln y=\ln\omega$$
$$+(2/n-2)\ln[n+4+(n-2)\omega]+\text{const.}$$

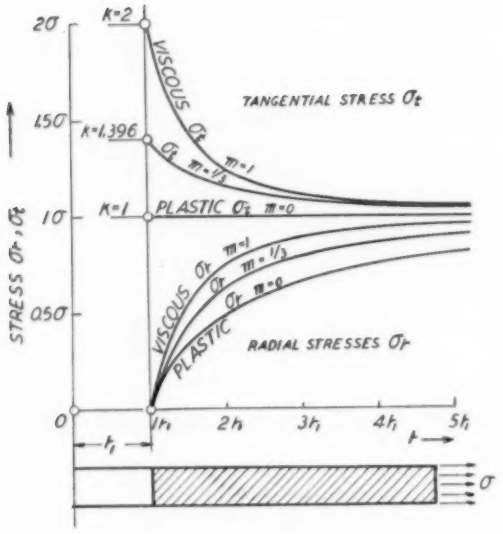


FIG. 9. Stresses around hole in disk in viscous ($m=1$), in creeping ($1>m>0$) and in plastic ($m=0$) substance. Stress concentration factor

$$K=(2/(1+3m))^{2m/(1-3m)}.$$

This equation, when u is written as abbreviation for the bracket can be rewritten as

$$y=c\omega u^{2/(n-2)} \tag{104}$$

and we see at the same time from (101) that

$$x=c(1+\omega)u^{2/(n-2)}. \tag{105}$$

The formulae (104) and (105) express the distribution of the stresses in the disk as the stresses σ_r and σ_t linearly depend on the variables x and y. Using (88) we obtain the stress expressed in parametric form:

$$\left.\begin{aligned}\sigma_r&=c(1-\omega)u^{2/(n-2)},\\ \sigma_t&=cu^{2/(n-2)},\end{aligned}\right\} \tag{106}$$

where $u=n+4+(n-2)\omega$. It remains finally to express the radius r or the variable $\rho=r/r_1$ as a function of the parameter ω. From (96) $dz/dy=w/y$. Remembering that $v=\ln y$, $dv=dy/y$ we see that

$$dz/dv=w \tag{107}$$

and that

$$dz/d\omega=(dz/dv)\cdot(dv/d\omega)=w\cdot dv/d\omega. \tag{108}$$

Since the right-hand terms w and $dv/d\omega$ in (108) are known functions of ω given by Eqs. (102) and (103), respectively, (108) is the differential equation for $z=\ln\rho$ as a function of the parameter ω. We obtain

$$\frac{dz}{d\omega}=-\frac{n+2+n\omega}{\omega[n+4+(n-2)\omega]}. \tag{109}$$

Since the boundary conditions for a disk having a hole $r=r_1$ and stretched by a uniform stress σ are $\rho=1$, $\sigma_r=0$ and $\rho=\infty$, $\sigma_r=\sigma_t=\sigma$, the corresponding values of the parameter $\omega=1-\sigma_r/\sigma_t$ must be: when $\rho=1$, $\omega=1$ and when $\rho=\infty$, $\omega=0$. Eq. (109) is therefore to be integrated under the condition: $\omega=1$, $z=\ln\rho=\ln 1=0$. The integral of (109) satisfying these latter conditions is:

$$\rho=\frac{r}{r_1}=\omega^{-(n+2)/(n+4)}\times\left[\frac{u}{2(n+1)}\right]^{-(4(n+1))/[(n-2)(n+4)]}. \tag{110}$$

The integration constant c in (106) is found from the second boundary condition that when $\omega=0$, $\sigma_t=\sigma$. This is the case when $c=(n+4)\sigma^{-2/(n-2)}$. The two Eqs. (106) together with (110) solve the problem completely.

The steady creep distribution of the stresses around the hole in the disk is given in parametric form by the following three equations:

$$\text{rad. stress } \sigma_r=(1-\omega)\sigma\left(1+\frac{n-2}{n+4}\omega\right)^{2/(n-2)}, \tag{111}$$

$$\text{tang. stress } \sigma_t=\sigma\left(1+\frac{n-2}{n+4}\cdot\omega\right)^{2/(n-2)}, \tag{112}$$

$$\left(\frac{r_1}{r}\right)^{(n+4)/(n+2)}=\omega\left[\frac{n+4+(n-2)\omega}{2(n+1)}\right]^{(4(n+1))/(n^2-4)}. \tag{113}$$

The factor of stress concentration k is found from (112) by taking for $\omega=1$

$$k=\sigma_t/\sigma=[2(n+1)/(n+4)]^{2/(n-2)} \tag{114}$$

or after introducing here the exponent m appearing in the creep speed law, this factor is found to be equal to

$$k=\left(\frac{2}{1+3m}\right)^{2m/(1-3m)}. \tag{115}$$

TABLE I. *Factor of stress concentration for creep and plastic substance.*

Exponent $m=1$	2/3	1/2	1/3	1/4	1/5	1/6	0
$n=(1/m)-1=0$	1/2	1	2	3	4	5	∞
Factor of stress concentration $k=2$	$(2/3)^{-4/3}$	$(4/5)^{-2}$	$e^{1/3}$	$(8/7)^2$	5/4	$(4/3)^{2/3}$	1
$k=2$	1.72	1.56	1.40	1.31	1.25	1.21	1
Substance: elastic or viscous	after steady creep						plastic

Formula (115) is valid for $0<m<1$. As $n=1/m-1$ Eqs. (111–114) are valid for $0<n<\infty$. The preceding equations must contain as special limiting cases the distributions of stress in a disk around a hole for a perfectly viscous and for a perfectly plastic substance. The first limiting case is obtained taking for the exponent m the value one. When $m=1$, $n=0$ we obtain from (113)

$$\frac{r_1^2}{r^2}=\frac{\omega}{2-\omega}$$

and from (111) and (112)

$$\sigma_r=\frac{2(1-\omega)\sigma}{2-\omega}, \quad \sigma_t=\frac{2\sigma}{2-\omega},$$

after eliminating ω we obtain

$$\sigma_r=\sigma(1-r_1^2/r^2), \quad \sigma_t=\sigma(1+r_1^2/r^2) \quad (116)$$

or the well-known formula for the stresses in a disk having a hole, valid in a viscous as well as in an elastic substance. When $m=1$ Eq. (115) gives the factor of stress concentration $k=2$ known for the elastic substance.

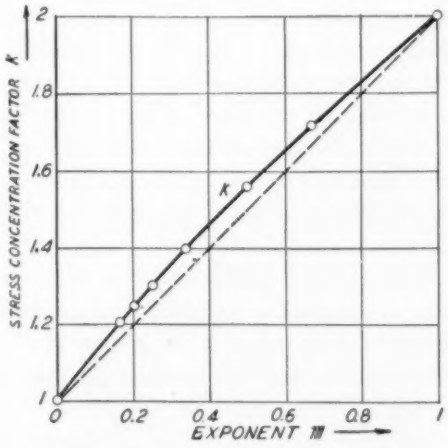


FIG. 10. Factor of stress concentration $K=(2/(1+3m))^{2m/(1-2m)}$ plotted *versus* exponent m of speed law $\tau=\tau_1(v/v_1)^m$ for disk with hole.

The second limiting case is given when $m=0$, $n=\infty$. From the Eqs. (111–115) one finds $\omega=r_1/r$, $\sigma_t=\sigma=$const. and $\sigma_r=(1-\omega)\sigma=\sigma(1-r_1/r)$. These are the well-known formulae for the stresses in a disk having a hole for a perfectly plastic substance assuming that the disk material is yielding uniformly under a constant maximum shearing stress $\tau=\sigma/2$. The factor k is in this case equal one.

Table I contains the values of the factor of stress concentration k computed with the formula (115) for a few values of the exponent m of the creep speed law.

The values of k are also plotted in Fig. 10 in function of the exponent m. The curve representing the values of $k=f(m)$ is not different much from the straight line connecting the points $m=0$, $k=1$ and $m=1$, $k=2$. A stress concentration factor in the steady state of creep can therefore be taken approximately as

$$k=1+m \quad (117)$$

under the assumptions as to the creep speed law. A few distributions of the stresses are shown in Fig. 9. One interesting case deserves perhaps to be mentioned especially. This latter occurs when the exponent approaches the value $m=\frac{1}{3}$ or n the value 2. The formula for k in this case becomes indefinite and can be written as

$$k=\lim_{n=2}[1+2(n+4)/2(n-2)]^{2/(n-2)}$$

or after writing here for $2/(n-2)=\alpha$ also as

$$k=\lim_{\alpha=\infty}\left(1+\frac{1}{3\alpha}\right)^{\alpha}=e^{\frac{1}{3}}=1.396. \quad (118)$$

The formulae (111–113) fail in this case, direct computation however furnishes when $n=2$ the rather simple functions

$$\sigma_r=\sigma(1-\omega)e^{\omega/3}, \quad \frac{r_1}{r}=\omega^{\frac{1}{3}}\cdot e^{(\omega-1)/3}. \quad (119)$$
$$\sigma_t=\sigma e^{\omega/3},$$

again valid for $0<\omega<1$. Taking $\omega=1$ the tangential stress at the bore is obtained equal to

$$\sigma_t=e^{\frac{1}{3}}\cdot\sigma=1.396\sigma \quad (120)$$

in agreement with the value of k just computed above in (118).

We note again that the formulae expressing the stresses σ_r and σ_t Eqs. (112), (113), (114) and the factor of stress concentration k Eq. (115) or (117) are also valid in the case of the strain dependent plastic flow in a thin disk having a hole, the material of which is strain-hardening according to the power function law

$$\tau=\tau^*(\gamma/\gamma^*)^m.$$

Notes on the Wetting of Filaments by Molten Metals

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THE following results of some experiments on the wetting of hot filaments by molten metals are presented in the hope that they may prove helpful to other experimenters who, like the author, may have searched the literature in vain for such information.

The object of the experiments was to determine which filament material was most desirable for use in evaporating a metal directly from the filament. In general if the molten metal adhered to the filament it would evaporate successfully unless its vapor pressure was too low at the highest safe temperature for the filament or unless it united with the filament to form a low melting point alloy.

The apparatus used consisted of a bell jar placed on a flat plate of heavy glass ground to fit; the joint sealed with mercury. The lead-in wires were introduced through a seal in the evacuating tube which in turn was sealed into the lower side of the flat glass plate by means of a ground joint sealed with de Khotinsky cement. Included in the system near the bell jar side of the trap was a discharge tube having two aluminum disk electrodes 24 mm in diameter spaced 12.8 cm apart. By applying the potential from a hand Tesla coil to one electrode, the other being grounded, and observing the form of the discharge it was possible to make a rough determination of the pressure in the system.

The metal to be evaporated was secured in the form of a wire of convenient size and wound around the filament to be tested. The wound filament was then formed into a spiral with short straight ends. These were inserted into connectors provided for them; pieces of glass as test strips were then placed above and below the filament at a distance of about 5 cm and the bell-jar lowered into place. When the potential from the Tesla coil failed to produce a discharge across the test gap the pressure was considered low enough for satisfactory evaporation.

A potential was applied to the filament and the current gradually increased until the metal melted and either adhered or dropped off. If it adhered, the current was further increased until the metal evaporated (as indicated by the coating of the strips of glass included in the system) or the filament burned out.

In Table I three results are indicated: (1) whether or not the metal when molten ad-

TABLE I.

	NICKEL 1452°–2900°C	PLATINUM 1755°–4300°C	MOLYBDENUM 2620°–3700°C
Aluminum 658.7°–1800°C	No experimental evidence.	1. Wets perfectly. 2. No success—filament burned out in spite of great care. 3. Metal probably united with filament to form low melting point alloy.	1. Wets perfectly. 2. Evaporates but requires careful control of temperature. 3. Not much apparent alloying.
Silver 960.5°–1950°C	1. Wets perfectly. 2. Evaporates well. 3. Probably alloys—deposited metal may contain some nickel.	No experimental evidence.	Melts and drops off.
Gold 1063°–2600°C	No experimental evidence.	No experimental evidence.	1. Melts and clings in globules. 2. Evaporates well. 3. No obvious effect of metal on filament
Copper 1083°–2310°C	1. Wets perfectly. 2. No success. 3. Copper probably alloys with nickel to form low melting point alloy.	1. Wets perfectly. 2. Evaporates fairly well. 3. No apparent interaction between metal and filament.	1. Melts and clings in droplets. 2. Evaporates well—though drops occasionally fall off. 3. No obvious interaction between copper and filament material.
Nickel 1452°–2900°C	Requires care but evaporates very successfully.	No experimental evidence.	1. Wets perfectly. 2. No success—filament burns out. 3. Nickel seems to unite with filament to form low melting point alloy.
Chromium 1615°–2200°C	Obviously impossible.	No experimental evidence.	No experimental evidence. (Metal not available in wire form.)
Platinum 1755°–4300°C	Obviously impossible.	No success as yet.	1. Wets perfectly. 2. Not much success—vapor pressure of metal too low at safe maximum filament temperature. 3. No obvious interaction between metal and filament.

	TANTALUM 2850°–5300°C	TUNGSTEN 3370°–5300°C
Aluminum 658.7°–1800°C	1. Wets perfectly. 2. Evaporates very well. 3. No obvious effect of metal on filament.	1. Wets perfectly. 2. Evaporates but requires some care. 3. Filament appears to have been affected slightly by metal.
Silver 960.5°–1950°C	Melts and drops off.	Melts and drops off.
Gold 1063°–2600°C	1. Melts and clings in globules. 2. No success—gold appears coated with a slag or scum. 3. Gold may have united with filament to form alloy.	1. Wets but hangs in globules. 2. Evaporates well. 3. No apparent interaction between metal and filament.
Copper 1083°–2310°C	1. Melts and clings in droplets. 2. Evaporates fairly well, though drops occasionally drop off. 3. No obvious interaction between copper and filament material.	Melts and drops off.
Nickel 1452°–2900°C	1. Wets perfectly. 2. No success—filament burns out. 3. Nickel seems to unite with filament to form low melting point alloy.	1. Wets perfectly. 2. Not successful—filament burns out easily. 3. Probably some interaction between metal and filament.
Chromium 1615°–2200°C	No experimental evidence. (Metal not available in wire form.)	Tried chromium plating filament with little success.
Platinum 1755°–4300°C	1. Melted, then sputtered and seemed to burn up. 2. No success—filament burned out. 3. Probably due to interaction between metal and filament.	1. Wets perfectly. 2. Requires care but is fairly successful. 3. No obvious interaction.

hered to the filament, (2) whether or not the metal evaporated successfully, and (3) whether or not there was obvious interaction between the molten metal and the filament material.

The filament material used is given at the head of each column, and the metals which were to be evaporated are listed in the column at the left. Under the name of each metal is given the melting point and the boiling point in degrees centigrade at atmospheric pressure.

The writer acknowledges with gratitude his indebtedness to Dr. P. H. Carr of the physics department who suggested the research problem and did much to encourage its continuation.

A Note on a Problem in Potential Theory

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(Received March 22, 1937)

Two methods are given for treating a new type of potential problem arising in a recent study of the author on the encroachment of water into an oil sand. The problem is that of finding the potential distributions in two regions of different "constants" ("conductivities"), separated by a moveable surface of unknown shape, this motion and shape to be determined by the conditions that each point of the interface should have a velocity proportional to the vector gradient of the potential at the point, and that the area swept out by the moving surface shall assume the "constant" appropriate to that of the encroaching side of the interface. The first method is a perturbation method, valid when the difference between the "constants" is small, and reduces the problem of determining the shape and motion of the interface and the potential distributions *simultaneously*, to a series of problems in which potential distributions are to be found on the sides of *known* interfaces, and the motion of the surface is to be found in a *known* (time varying) potential field. This same type of reduction is effected also in the second method, which is one of direct successive approximations, but which may be used for any values of the potential "constants." The perturbation theory is carried through explicitly to the third order for the case of a one dimensional system, and agrees to that order with the exact results which were derived for this case in a previous paper. The method of successive approximations is carried out for this same case through the second approximation for a ratio of the "conductivity" of the interior side of the interface to that on the exterior side equal to 5, and through the third approximation for the ratio 1/5. In the former, the calculations give a rapid monotonic convergence to the correct solution, and in the latter the convergence is oscillatory, and only slightly less rapid. These behaviors are shown to be generally true for the linear problem, in the course of the construction of an analytic convergence proof for the one dimensional problem. The convergence proof of the method of successive approximations is also given for the case of radial symmetry.

IN a recent paper in this journal,[1] the author, in discussing the encroachment of water into an oil sand, outlined a new type of potential problem. This problem was formulated as follows: Find the potential functions Φ_1 between a fixed surface S_w and the moveable surface $F(x, y, z, t)=0$, and Φ_2 between $F(x, y, z, t)=0$ and the fixed surface S_e, such that[2]

$$\left.\begin{aligned} &\Phi_1=\Psi_w(x, y, z, t) \text{ on } S_w,\\ &\Phi_2=\Psi_e(x, y, z, t) \text{ on } S_e,\\ &\left.\begin{aligned} &\Phi_1=\Phi_2,\\ &c_1\partial\Phi_1/\partial n=c_2\partial\Phi_2/\partial n \end{aligned}\right\} \text{ on } F(x, y, z, t)=0, \end{aligned}\right\} \quad (1)$$

where

$$\partial F/\partial t-c_{1,2}\nabla\Phi_{1,2}\cdot\nabla F=0; \quad F(x, y, z, 0)=S_i,$$

n being the normal to F. The system is diagrammatically indicated in Fig. 1, where the moveable surface F is denoted by S_i, its initial form being taken for convenience as S_e.

This problem was new in the sense that instead of the normal potential problem in which the potentials are to be found on the two sides of a *fixed* interface separating two regions of different "constants," that in question involves a *moving* interface, the motion and shape of which must be determined *simultaneously* with the potential distributions. It corresponds physically, in the problem where it arose, to a system in which oil, of viscosity μ_1, originally occupying the sand between S_w and S_e, is displaced by the water, of viscosity μ_2, encroaching into the oil sand across the surface S_e, the pressures at S_w and S_e being considered as known. The surface S_i is then the continuously changing water-oil interface, and the constants c are the reciprocals of the liquid viscosity. The potential functions Φ are the velocity potentials kp of the liquid (neglecting effects of gravity), where k is the permeability of the sand, taken as uniform, and p is the fluid pressure.

[1] M. Muskat, Physics **5**, 250 (1934).
[2] The time is to be considered as a parameter when it occurs in the potential distributions, as the changes in the potential distributions due to changes in the boundary values or positions of the interface are to be considered as taking place instantaneously.

No general treatment of this problem was given in the above mentioned paper. Rather, rigorous solutions were presented only for the three simple cases of linear, radial and spherical flow, where the shape of the interface S_i was obvious *a priori* from the geometry of the systems. For the more general cases, the discussion was based upon the "zeroth" approximation to the correct solution, in which the differences between c_1 and c_2 were neglected, and the shape and motion of S_i were taken to be those of a line or surface of fluid particles in a homogeneous system of the same geometry, leaving S_e at the same instant as the real passage of the water across S_e was supposed to take place (cf. Eq. (4) below).

In this note two methods will be presented for reducing the problem of the unknown moving interface to a series of potential problems, to be solved successively, in each of which the shape and motion of the interface are to be considered as known from a previous approximation. The time will enter in the analysis of the potential problems only as a parameter defining the shape and position of the interface. The first method will consist of a formal perturbation theory, applicable to cases where the difference between the constants c_1, c_2 may be taken as small. The second will be a method of direct successive approximations valid for any values of c_1 and c_2. Although no general proofs have been developed regarding the convergence of the methods, the correctness of the perturbation theory to the third order and the rapid convergence of the successive approximation method in the simple case of linear flow, where the exact solution was derived in the paper cited above, will be demonstrated by the actual calculations.

Perturbation Theory for $|c_2-c_1| = |\eta| \ll c_{1,2}$

Taking for convenience $c_1=1$, and $c_2=c=1+\eta$, the perturbation theory consists simply in considering the functions Φ_1, Φ_2, and F to be expanded in power series in the small parameter η. Thus:

$$\left.\begin{aligned}\Phi_1&=\Phi_{10}+\eta\Phi_{11}+\eta^2\Phi_{12}+\cdots,\\ \Phi_2&=\Phi_{20}+\eta\Phi_{21}+\eta^2\Phi_{22}+\cdots,\\ F&=F_0+\eta F_1+\eta^2F_2+\cdots,\end{aligned}\right\} \qquad (2)$$

where all the Φ's are potential functions. Applying these to Eq. (1) and denoting by $\Phi_{1,2}^{(m)}$, $F^{(m)}$ the resultant values of $\Phi_{1,2}$, F to the mth approximation, the general problem in the mth approximation may be expressed as that of finding the solutions to the equations:

$$\left.\begin{aligned}&\nabla^2\Phi_1^{(m)}=\nabla^2\Phi_2^{(m)}=0,\\ &\Phi_1^{(m)}=\Psi_w \text{ on } S_w;\quad \Phi_2^{(m)}=\Psi_e \text{ on } S_e,\\ &\Phi_1^{(m)}=\Phi_2^{(m)}\\ &\frac{\partial\Phi_1^{(m)}}{\partial n}=(1+\eta)\frac{\partial\Phi_2^{(m)}}{\partial n}\cong\frac{\partial\Phi_2^{(m)}}{\partial n}+\eta\frac{\partial\Phi_2^{(m-1)}}{\partial n}\\ &\qquad\qquad \text{on } F^{(m-1)}(x,y,z,t)=0,\\ &\frac{\partial F^{(m)}}{\partial t}=\nabla\Phi_1^{(m)}\cdot\nabla F^{(m)};\quad F^{(m)}(x,y,z,0)=S_e\end{aligned}\right\} \qquad (3)$$

or
$$\partial F_m/\partial t=\sum_{j+k=m}\nabla\Phi_{1j}\cdot\nabla F_k.$$

Carrying this formulation through explicitly for the first two approximations, we have:

Zeroth approximation:

$$\left.\begin{aligned}&\nabla^2\Phi_{10}=\nabla^2\Phi_{20}=0,\\ &\Phi_{10}\equiv\Phi_{20}=\begin{cases}\Psi_w \text{ on } S_w,\\ \Psi_e \text{ on } S_e,\end{cases}\\ &\partial F_0/\partial t=\nabla\Phi_{10}\cdot\nabla F_0;\quad F_0(x,y,z,0)=S_e.\end{aligned}\right\} \qquad (4)$$

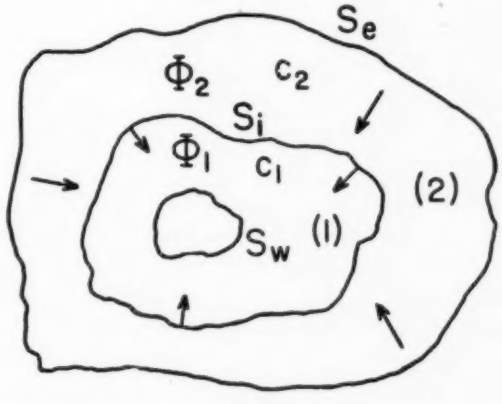

Fig. 1.

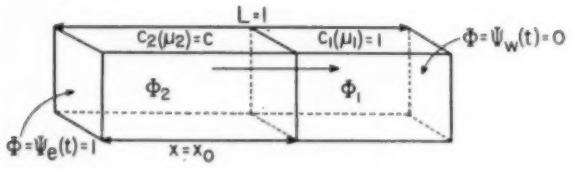

FIG. 2. Diagrammatic representation of a linear encroachment system.

First approximation:

$$\left.\begin{aligned}
&\nabla^2\Phi_{11}=\nabla^2\Phi_{21}=0,\\
&\Phi_{11}=0 \text{ on } S_w;\quad \Phi_{21}=0 \text{ on } S_e,\\
&\left.\begin{aligned}\Phi_{11}&=\Phi_{21}\\ \partial\Phi_{11}/\partial n&=\partial\Phi_{21}/\partial n+\partial\Phi_{20}/\partial n\end{aligned}\right\}\text{ on } F_0,\\
&\partial F_1/\partial t=\nabla\Phi_{10}\cdot\nabla F_1+\nabla\Phi_{11}\cdot\nabla F_0,\\
&F^{(1)}=F_0+\eta F_1\rightarrow S_e, \text{ at } t=0.
\end{aligned}\right\}\quad(5)$$

Second approximation:

$$\left.\begin{aligned}
&\nabla^2\Phi_{12}=\nabla^2\Phi_{22}=0,\\
&\Phi_{12}=0 \text{ on } S_w;\quad \Phi_{22}=0 \text{ on } S_e,\\
&\left.\begin{aligned}\Phi_{12}&=\Phi_{22}+(1/\eta)(\Phi_{21}-\Phi_{11})\\ \frac{\partial\Phi_{12}}{\partial n}&=\frac{\partial\Phi_{22}}{\partial n}+\frac{\partial\Phi_{21}}{\partial n}\\ &\quad+\frac{1}{\eta}\left(\frac{\partial\Phi_{20}}{\partial n}+\frac{\partial\Phi_{21}}{\partial n}-\frac{\partial\Phi_{11}}{\partial n}\right)\end{aligned}\right\}\text{ on } F^{(1)},\\
&\partial F_2/\partial t=\nabla\Phi_{10}\cdot\nabla F_2+\nabla\Phi_{11}\cdot\nabla F_1+\nabla\Phi_{12}\cdot\nabla F_0,\\
&F^{(2)}=F_0+\eta F_1+\eta^2F_2\rightarrow S_e, \text{ at } t=0,\\
&\text{etc.}
\end{aligned}\right\}\quad(6)$$

THE LINEAR PROBLEM

For the one dimensional case, diagrammatically represented by Fig. 2, where the interface in the whole system, of unit length, is defined by its distance x_0 from the inflow face S_e, with a unit potential difference between S_e and S_w, i.e.: $\Psi_w=0$, $\Psi_e=1$, the above equations lead to the following analysis:

Zeroth approximation:

$$\left.\begin{aligned}
&\Phi_{10}\equiv\Phi_{20}=1-x,\\
&\partial F_0/\partial t=-\partial F_0/\partial x;\quad F_0=t-x_0=0.
\end{aligned}\right\}\quad(7)$$

First approximation:

$$\Phi_{11}=a_{11}(x-1);\quad \Phi_{21}=a_{21}x,$$

where

$$a_{11}(x_0-1)=a_{21}x_0;\quad a_{11}=a_{21}-1,$$

so that $a_{11}=-x_0=-t$; $\quad a_{21}=1-x_0=1-t$.

Further

$$\left.\begin{aligned}
&\partial F_1/\partial t=-\partial F_1/\partial x-a_{11};\quad F_1=t-x+t^2/2.\\
&\text{Hence}\\
&(t-x^{(1)})(1+\eta)+\eta t^2/2=0;\\
&x^{(1)}=t+\eta t^2/2(1+\eta).
\end{aligned}\right\}\quad(8)$$

Second approximation:

$$\Phi_{12}=a_{12}(x-1);\quad \Phi_{22}=a_{22}x,$$

$$\begin{aligned}
a_{12}(x^{(1)}-1)&=a_{22}x^{(1)}+(1/\eta)\{a_{21}x^{(1)}\\
&\quad-a_{11}(x^{(1)}-1)\}=a_{22}x^{(1)}+t^2/2(1+\eta),\\
a_{12}&=a_{22}+a_{21}+(1/\eta)\{-1+a_{21}-a_{11}\}\\
&=a_{22}+1-t.
\end{aligned}$$

Hence

$$a_{12}=t(1-t)-\frac{t^2}{2(1+\eta)}\{1-\eta(1-t)\},$$

$$a_{22}=-(1-t)^2-\frac{t^2}{2(1+\eta)}\{1-\eta(1-t)\}.$$

Further,

$$\partial F_2/\partial t=-\partial F_2/\partial x-a_{11}-a_{12},$$

so that

$$\left.\begin{aligned}
&F_2=t-x+\frac{t^3}{3}+\frac{t^3(1-\eta)}{6(1+\eta)}+\frac{\eta t^4}{8(1+\eta)}.\\
&\text{Finally}\\
&(t-x)(1+\eta+\eta^2)+\frac{\eta t^2}{2}+\frac{\eta^2t^3}{3}+\frac{\eta^2(1-\eta)t^3}{6(1+\eta)}\\
&\qquad+\frac{\eta^3t^4}{8(1+\eta)}=0,\\
&\text{and}\\
&x^{(2)}=t+\eta t^2/2-\eta^2t^2/2+\eta^2t^3/2+0(\eta^3).
\end{aligned}\right\}\quad(9)$$

Third approximation:

Here the extensions of Eq. (6), reduced to the problem of linear flow, are:

$$\left.\begin{aligned}
&\Phi_{13}=a_{13}(x-1); \quad \Phi_{23}=a_{23}x,\\
&\Phi_{13}=\Phi_{23}+(1/\eta)(\Phi_{22}-\Phi_{12})\\
&\qquad\qquad +(1/\eta^2)(\Phi_{21}-\Phi_{11}) \text{ on } x^{(2)},\\
&\frac{\partial\Phi_{13}}{\partial x}=\frac{\partial\Phi_{23}}{\partial x}+\frac{\partial\Phi_{22}}{\partial x}+\frac{1}{\eta}\left(\frac{\partial\Phi_{22}}{\partial x}+\frac{\partial\Phi_{21}}{\partial x}-\frac{\partial\Phi_{12}}{\partial x}\right)\\
&\qquad\qquad +\frac{1}{\eta^2}\left\{\frac{\partial\Phi_{21}}{\partial x}+\frac{\partial\Phi_{20}}{\partial x}-\frac{\partial\Phi_{11}}{\partial x}\right\},\\
&\frac{\partial F_3}{\partial t}=-\frac{\partial F_3}{\partial x}-a_{11}-a_{12}-a_{13},
\end{aligned}\right\} \tag{10}$$

where, as F_3 will enter into $x^{(3)}$ with the coefficient η^3, only the zero-order terms need be retained in a_{12} and a_{13}. Solving these in the same manner as those for the second approximation, it is found that to the zero order

$$a_{13}=-t+2t^2-2t^3,$$

so that

$$F_3=t-x+t^2/2-t^3/6+t^4/2,$$

and finally

$$x^{(3)}=t+\frac{\eta t^2}{2}+\frac{\eta^2 t^2}{2}(t-1)+\frac{\eta^3 t^2}{8}(4-8t+5t^2). \tag{11}$$

That these approximations of Eqs. (8), (9), and (11) are correct to the orders indicated, may be readily verified by expanding the exact solution for the one dimensional problem, namely:

$$x_0=\frac{\eta+1}{\eta}\left\{1-\left(1-\frac{2\eta t}{\eta+1}\right)^{\frac{1}{2}}\right\}. \tag{12}$$[3]

Although the actual determination of the *shape* of the interface F has been unnecessary in the above perturbation calculations, just as in the case of the rigorous analysis of the linear problem, the analytical content of the perturbation method is fully illustrated by the treatment of the one dimensional case. The only essential difference between the one and the two and three dimensional problems is that in the latter the partial differential equations for F_m, the last of Eqs. (3), will involve two or three of the coordinates (x, y, z) rather than just x. In all cases, however, the program contained in Eq. (3) gives an explicit reduction of the original problem of determining the shape and motion of the interface and the potential distributions *simultaneously*, to the series of problems in which these parts of the complete solution may be derived by *independent* analytic processes.

[3] Reference 1, Eq. (11), where ϵ is to be replaced by $1/(1+\eta)$ and x_0 by $1-x$.

The Method of Successive Approximations

The method of successive approximations proposed here begins with the same zero approximation as the above perturbation theory. It may be formulated as follows, where the functions Φ_{1m}, Φ_{2m} and F_m represent here the *complete* solutions to the mth approximation.

$$\left.\begin{aligned}
&\nabla^2\Phi_{1m}=\nabla^2\Phi_{2m}=0,\\
&\Phi_{1m}=\Psi_w \text{ on } S_w; \quad \Phi_{2m}=\Psi_e \text{ on } S_e,\\
&\left.\begin{aligned}\Phi_{1m}&=\Phi_{2m}\\ c_1\partial\Phi_{1m}/\partial n&=c_2\partial\Phi_{2m}/\partial n\end{aligned}\right\} \text{ on } F_{m-1},\\
&\partial F_m/\partial t=-\bar{v}_m\cdot\nabla F_m; \quad F_m(x, y, z, 0)=S_e,
\end{aligned}\right\} \tag{13}$$

where $\bar{v}_m$ is the velocity vector $-c_1\nabla\Phi_{1m}$ at F_m.

Thus the first approximation involves the finding of the strict potential distribution for a system with constants c_1, c_2, separated by the *known* moving interface F_0 given by the zeroth approximation. The motion of a line of fluid particles in a homogeneous medium of constant c_1, but with the *velocity field* corresponding to the new first approximation potential distribution, gives the new interface F_1. This same process is repeated with F_1 taken as the known moving interface, to give the second approximations Φ_{12}, Φ_{22}, and then F_2, etc. Although the physical interpretation of this approximation method would lead one to expect that the process will actually converge to the correct solution, an analytical convergence proof has not as yet been constructed except for the cases which had previously been solved rigorously. Nevertheless the application of the method to these simple cases does show that at least there the numerical convergence is quite satisfactory, as well as attainable in an analytical sense.

The Linear Problem

Using the same notation as in Fig. 1, except that the constant $c_2=c$ for region (2), which really represents the ratio c_2/c_1, is no longer $1+\eta$, we have:

Zeroth approximation:

$$\Phi_{10}\equiv\Phi_{20}=1-x;\quad F_0=t-x=0,\qquad x^{(0)}=t. \tag{14}$$

First approximation:

$$\left.\begin{aligned}&\Phi_{11}=a_{11}(x-1);\quad \Phi_{21}=a_{21}x+1,\\ &a_{11}(x^{(0)}-1)=a_{21}x^{(0)}+1;\quad a_{11}=ca_{21},\end{aligned}\right\} \tag{15}$$

so that

$$\left.\begin{aligned}&a_{11}=-v_1=\frac{-1}{1+\alpha t};\\ &a_{21}=\frac{-(\alpha+1)}{1+\alpha t};\quad \alpha=\frac{1-c}{c},\\ &\frac{\partial F_1}{\partial t}=\frac{-1}{1+\alpha t}\frac{\partial F_1}{\partial x};\quad F_1=x-\int_0^t\frac{dt}{1+\alpha t}=0,\\ &x^{(1)}=(1/\alpha)\log(1+\alpha t).\end{aligned}\right\} \tag{16}$$

Second approximation:

$$\left.\begin{aligned}&\Phi_{12}=a_{12}(x-1);\quad \Phi_{22}=a_{22}x+1,\\ &a_{12}(x^{(1)}-1)=a_{22}x^{(1)}+1;\quad a_{12}=ca_{22},\end{aligned}\right\} \tag{17}$$

so that

$$a_{12}=-v_2=-1/(1+\log(1+\alpha t)).$$

Hence,

$$\frac{\partial F_2}{\partial t}=\frac{-1}{1+\log(1+\alpha t)}\frac{\partial F_2}{\partial x},$$

which gives

$$\left.\begin{aligned}x^{(2)}&=\int_0^t\frac{dt}{1+\log(1+\alpha t)}\\ &=\frac{1}{\alpha e}\{Ei(-u)-Ei(1)\},\\ \text{where}\quad &-u=1+\log(1+\alpha t).\end{aligned}\right\} \tag{18}$$

Carrying through the same procedure so as to obtain the third approximation, it is found that

$$x^{(3)}=\int_0^t\frac{dt}{1+\alpha x^{(2)}}, \tag{19}$$

and in fact, for the nth approximation,

$$x^{(n)}=\int_0^t\frac{dt}{1+\alpha x^{(n-1)}}, \tag{20}$$

which may also be derived directly on noting that the velocity in a linear system separated at $x^{(n-1)}$ into regions of "constants" 1 and c is simply the integrand of Eq. (20).

These formulas have been applied numerically to the second approximation for $c=5$, $(\alpha=-0.8)$ and the third approximation—this last by graphically integrating Eq. (19)—for $c=\frac{1}{5}$, $(\alpha=4)$. The results are shown in Fig. 3. It will be seen that for $c=5$ the second approximation already gives a very close approximation to the correct motion of the interface. In the case $c=\frac{1}{5}$, the convergence is not quite so rapid. However, in view of the large difference between the correct and zeroth approximation solutions, the convergence may still be considered to be quite satisfactory.

The monotonic convergence for $c=5(>1)$ and oscillatory convergence for $c=\frac{1}{5}(<1)$ are general consequences of Eq. (20). In fact, noting that the correct solution $x(t)$ obeys Eq. (20) in the limit where $x^{(n-1)}=x^{(n)}=x$, namely:

$$x(t)=\int_0^t\frac{dt}{1+\alpha x}, \tag{21}$$

it readily follows that for

$$\left.\begin{aligned}&\alpha<0,\quad \text{then}\quad x^{(n)}(t)\leqq x(t),\\ &\qquad\qquad\text{as}\quad x^{(n-1)}(t)\leqq x(t),\\ &\text{while for}\\ &\alpha>0,\qquad x^{(n)}(t)\leqq x(t),\\ &\qquad\qquad\text{as}\quad x^{(n-1)}(t)\geqq x(t).\end{aligned}\right\} \tag{22}$$

Observing, furthermore, that

$$\left.\begin{aligned}&\alpha<0,\quad x^{(n)}(t)\leqq x^{(n-1)}(t),\\ &\qquad\qquad\text{as}\quad x^{(n-1)}(t)\leqq x^{(n-2)}(t),\\ &\alpha>0,\quad x^{(n)}(t)\leqq x^{(n-1)}(t),\\ &\qquad\qquad\text{as}\quad x^{(n-1)}(t)\geqq x^{(n-2)}(t),\end{aligned}\right\} \tag{23}$$

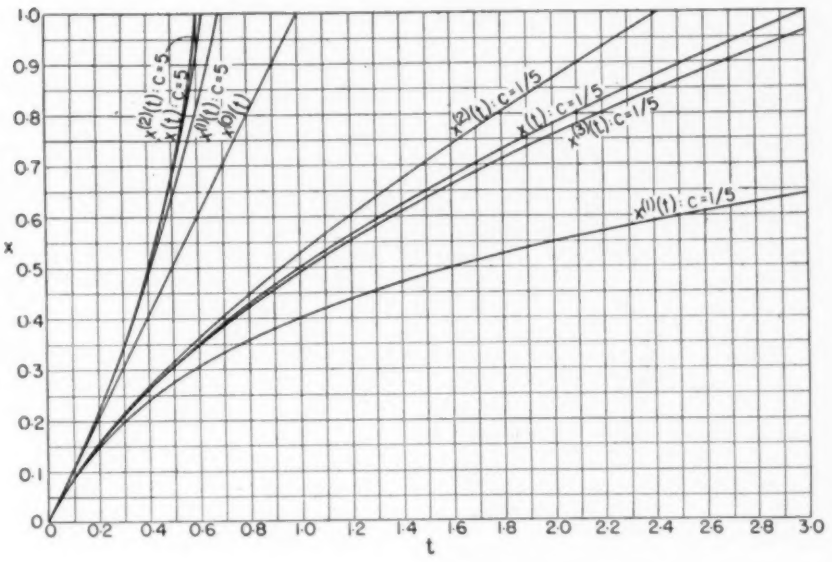

FIG. 3. The advance of the interface in a linear encroachment system as a function of the time, t, as calculated by the method of successive approximations. x = distance of the interface from initial position. $x(t)$ is position as given by exact solution. $x^{(n)}(t)$ is position as given by the n^{th} approximation. c = ratio of "potential constant" on back side of interface to that on front side.

it follows that since for $\alpha<0$, $x(t)>x^{(1)}(t)>x^{(0)}(t)$, $x^{(n)}(t)$ must increase monotonically and converge to $x(t)$ (or at least to values less than $x(t)$). Similarly, since the inequalities of Eq. (23) can also be applied to the elements of the sequence $x^{(n)}$, $x^{(n-2)}$, $x^{(n-4)}$, it follows for $\alpha>0$ that since $x^{(0)}>x^{(2)}$:

$$\left.\begin{aligned} &x^{(0)}>x^{(2)}>x^{(4)}>\cdots x(t);\\ &x^{(1)}<x^{(3)}<x^{(5)}<\cdots x(t), \end{aligned}\right\} \tag{24}$$

thus proving the convergence to $x(t)$ of both the odd and even sequences of the approximations.

THE RADIAL PROBLEM (CF. FIG. 4)

The equation for the interface in the nth approximation is here evidently

$$\partial F_n/\partial t=-v_n\partial F_n/\partial r, \tag{25}$$

and from elementary potential theory it is readily found that:

$$\left.\begin{aligned} v_n=-\frac{\Delta\Phi}{r[\log(r_e/r_w)+\alpha\log(r_e/r_0^{(n-1)})]};\\ \alpha=1/c-1, \end{aligned}\right\} \tag{26}$$

$r_0^{(n-1)}(t)$ giving the position of the interface to the $(n-1)$st approximation.

Taking the function F_n in the form

$$F_n=r-r_0^{(n)}(t)=0, \tag{27}$$

Eq. (25) becomes

$$\left.\begin{aligned} \frac{\partial r_0^{(n)}}{\partial t}=\frac{-\Delta\Phi}{r_0^{(n)}[\log(r_e/r_w)+\alpha\log(r_e/r_0^{(n-1)})]},\\ \text{so that}\\ \left(\frac{r_0^{(n)}}{r_e}\right)^2=1\\ -\frac{4\Delta\Phi}{r_e^2}\int_0^t\frac{dt}{\log r_e^2/r_w^2+\alpha\log(r_e/r_0^{(n-1)}(t))^2}. \end{aligned}\right\} \tag{28}$$

Furthermore, the exact solution $r_0(t)$ satisfies the equation

$$\left(\frac{r_0}{r_e}\right)^2=1-\frac{4\Delta\Phi}{r_e^2}\int_0^t\frac{dt}{\log r_e^2/r_w^2+\alpha\log(r_e^2/r_0^2)}. \tag{29}$$[4]

From Eqs. (28) and (29) it follows that for

$$\left.\begin{aligned} &\alpha<0: r_0^{(n)}(t)\leqq r_0(t), \quad \text{as} \quad r_0^{(n-1)}(t)\leqq r_0(t),\\ &\text{while for}\\ &\alpha>0: r_0^{(n)}(t)\leqq r_0(t), \quad \text{as} \quad r_0^{(n-1)}(t)\geqq r_0(t). \end{aligned}\right\} \tag{30}$$

[4] M. Muskat, reference 1, Eq. (18).

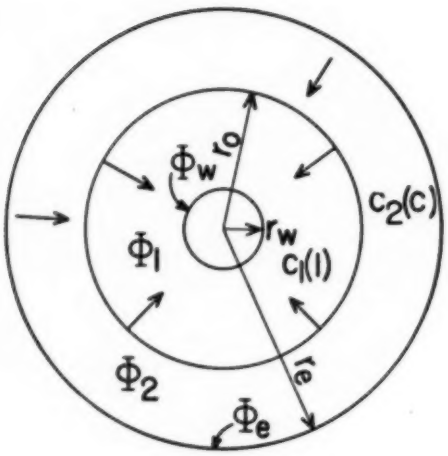

FIG. 4. Diagrammatic representation of a radial encroachment system.

And from Eq. (28), one may readily show that for:

$$\left.\begin{aligned} &\alpha<0 : r_0^{(n)}(t)\leqq r_0^{(n-1)}(t), \\ &\qquad\qquad \text{as} \quad r_0^{(n-1)}(t)\leqq r_0^{(n-2)}(t), \\ &\text{while for} \\ &\alpha>0 : r_0^{(n)}(t)\leqq r_0^{(n-1)}(t), \\ &\qquad\qquad \text{as} \quad r_0^{(n-1)}(t)\geqq r_0^{(n-2)}(t). \end{aligned}\right\} \tag{31}$$

The convergence of the sequences of approximations then follows by choosing[5] for $\alpha<0$, zero approximation functions for which either $r_0^{(1)}<r_0^{(0)}$ if $r_0^{(0)}>r_0(t)$ or $r_0^{(1)}>r_0^{(0)}$ if $r_0^{(0)}<r_0(t)$, and for $\alpha>0$, zero approximations for which $r_0^{(2)}<r_0^{(0)}$ if $r_0^{(0)}>r_0(t)$, and $r_0^{(2)}>r_0^{(0)}$ if $r_0^{(0)}<r_0(t)$. For the zero approximation as given by Eq. (28) ($\alpha=0$), it follows from Eqs. (28) and (29) that $r_0^{(0)}>r_0(t)$ and $r_0^{(1)}<r_0^{(0)}$ for $\alpha<0$, and $r_0^{(0)}<r_0(t)$ and $r_0^{(2)}>r_0^{(0)}$ for $\alpha>0$, thus giving the conditions for the actual convergence of the resulting sequences.

In an exactly similar manner one may demonstrate the convergence of the method given here in the problem possessing spherical symmetry, corresponding hydrodynamically to that of spherical encroachment.

Although here, too, the method of successive approximations has been illustrated by examples in which the shapes of the interface have been known a priori, the determination of this shape will, in the general problem, involve no analytical processes other than those already occurring in the treatment of the linear and radial cases. The determination of the potential distributions in the various approximations will again reduce to potential problems with *known* surfaces of discontinuity, the time entering only as parameters. The shape and motion of the interface will be given by the solution of the same partial differential equation, such as the last of Eq. (15), except that it will involve 3 or 4 of the independent variables (x, y, z, t) rather than just the two (x, t). Indeed, the solutions of these individual problems will in general be very difficult and perhaps even be intractable, from a practical point of view,[6] but at least they will all be of a kind that may be considered as part of known analytic methods.

The writer is indebted to Dr. L. J. Peters for helpful discussions of the problem treated here and to the executives of the Gulf Research & Development Company for permission to publish this paper.

[5] Insofar as the problem is reduced to the successive application of Eq. (28), one may choose for the zero approximation any function $r_0^{(0)}(t)$, the choice affecting only the rapidity of the convergence once the proper inequality with respect to $r_0^{(1)}$ or $r_0^{(2)}$ is satisfied.

[6] In fact, if one is willing to use graphical or numerical methods, it will probably be easier to attack the original problem directly by the following step-wise integration process. First, Φ_1 and Φ_2 would be computed (graphically or numerically) for the initial position of S_i. The velocity vectors $-c\nabla\Phi$ would then be constructed over S_i, the terminae of which, with a proper scale, will give the position of S_i at a short time interval after the initial instant. For this new position of S_i, Φ_1 and Φ_2 would again be computed, the velocity vectors drawn on S_i, etc.

Heat Flow in an Infinite Solid Bounded Internally by a Cylinder

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The problem of the flow of heat through a long tunnel wall in an infinite solid has been treated and the results for the rate of flow and total heat flow have been expressed in a convenient form for numerical computations. This general problem has become of interest in connection with the cooling of deep mines.

IN deep mining operations it has become increasingly important to cool the tunnels to a point where the temperature is such that men can work efficiently. In newly excavated tunnels the rock temperatures sometimes run from 120°F to 140°F and in order to reduce the temperatures so that work can be carried on efficiently in a reasonable time it is necessary to know the rate at which heat flows through the tunnel walls in the course of time.

In order to ascertain the heat flow under actual conditions it is necessary to have the solution of the following heat flow problem: Given an infinite solid of uniform characteristics bounded internally by a very long cylinder of radius R. At time $t=0$, the temperature through the solid will be a function $f(r)$ of the radial coordinate alone, i.e. $u=f(r)$ for $r>R$ and $t=0$. In practice deep borings have shown that the initial temperature distribution is essentially constant. Since the wall temperature may vary in the course of time it will be considered as an arbitrary function of the time for the present, i.e. $u=u_s(t)$ for $r=R$. A heat flow problem of this type has been considered by J. W. Nicholson[1] but the formulae are not very convenient for numerical computations.

The mathematical problem reduces to that of finding a solution of the heat equation

$$\partial u/\partial t=a(\partial^2 u/\partial x^2+\partial^2 u/\partial y^2+\partial^2 u/\partial z^2), \quad (1)$$

which satisfies the boundary and initial conditions specified above, where a is the diffusivity of the solid material. Since the temperature does not vary along the length of the cylinder, it may be shown[2] that a solution of Eq. (1) in cylindrical coordinates which satisfies the boundary conditions laid down above is given by

$$u(r,\theta,t)=\int_0^{2\pi}d\theta'\int_R^\infty g(r,\theta;r',\theta';t)f(r')r'dr' + a\int_0^t d\tau\int_0^{2\pi}u_s(R,\theta',\tau) \times\left\{\frac{\partial g}{\partial r'}(r,\theta;r';t-\tau)\right\}_{r'=R}Rd\theta' \quad (2)$$

provided the proper Green's function $g(r,\theta;r',\theta';t-\tau)$ can be found. The proper Green's function is that solution of the heat equation which gives the temperature at any point r', θ', at time t, due to a line source of unit strength located at the point r, θ at time τ, the solid being initially at zero temperature and the surface being kept at zero temperature. Therefore, to obtain the Green's function for this case, it is convenient to start with the solution of the heat equation representing the temperature from a line source of unit strength located at the point r, θ, namely

$$\frac{1}{4\pi at}\exp\frac{-r^2+r'^2-2rr'\cos(\theta-\theta')}{4at}. \quad (3)$$

In order to satisfy the condition that Green's function must be zero at $r'=R$ it is convenient to expand the above function in the Fourier series

$$\frac{1}{4\pi at}\exp -r^2+r'^2-2rr'\cos(\theta-\theta') =\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\cos n(\theta-\theta')\int_0^\infty se^{-ats^2}J_n(sr')J_n(sr)ds, \quad (4)$$

[1] Proc. Roy. Soc. **100**, 226 (1921).
[2] See Carslaw, *The Conduction of Heat*, p. 172.

where $J_n(x)$ is Bessel's function of order n. For future application of this series it will be found desirable to express the above integral as a contour integral in the complex z plane. This may be done by expressing one of the Bessel functions in terms of Hankel functions of the first kind. As is well known the Hankel functions $H_n^{(1)}(x)$ and $H_n^{(2)}(x)$ are also fundamental solutions of Bessel's equation. The relation[3] between $J_n(x)$ and $H_n^{(1)}(x)$ is

$$2J_n(x)=H_n^{(1)}(x)-e^{n\pi i}H_n^{(1)}(-x). \qquad (5)$$

With this relation the integral in Eq. (4) becomes

$$\int_0^\infty e^{-\alpha t s^2}J_n(sr')J_n(sr)sds = \tfrac{1}{2}\int_{-\infty}^{+\infty} e^{-\alpha t s^2}J_n(sr')H_n^{(1)}(sr)sds.$$

Consider next the integral

$$\int e^{-\alpha t z^2}J_n(zr')H_n^{(1)}(zr)zdz$$

taken over the contour indicated in Fig. 1. Since the integral has no poles in the positive half-plane the integral around the complete contour is zero, hence

$$\int_{-\rho}^{+\rho} e^{-\alpha t s^2}J_n(sr')H_n^{(1)}(sr)sds+\int_C e^{-\alpha t z^2}J_n(zr') \times H_n^{(1)}(zr)zdz+\text{integral over arcs}=0.$$

It may readily be shown that if $r>r'$ and the argument of z on the right-hand end of C is such that $0\leq\arg z<\pi/4$ and at the left-hand $(3\pi/4)<\arg z\leq\pi$ then the integral over the circular arcs vanish as $\rho\to\infty$ so that understanding C to mean the path with the above characteristics

$$\int_{-\infty}^{+\infty} e^{-\alpha t s^2}J_n(sr')H_n^{(1)}(sr)sds = -\int_C e^{-\alpha t z^2}J_n(zr')H_n^{(1)}(zr)zdz, \quad r>r'. \qquad (6)$$

For the case when $r<r'$ we need only interchange r and r' in the integral on the right.

[3] Watson, *Bessel's Functions*, p. 75.

The series in Eq. (4) thus reduces to

$$-\frac{1}{4\pi}\sum_{n=-\infty}^{+\infty}\cos n(\theta-\theta')\int_C e^{-\alpha t z^2}J_n(zr')H_n^{(1)}(zr)zdz, \quad r>r'. \qquad (7)$$

A Green's function which vanishes for $r'=R$ can be obtained by adding the solution

$$\frac{1}{4\pi}\sum_{n=-\infty}^{+\infty}\cos n(\theta-\theta')\int_C A(z)e^{-\alpha t z^2}J_n(zr')H_n^{(1)}(zr)zdz \qquad (8)$$

to solution (7), the A being determined so as to make the sum vanish at $r'=R$. It is readily seen

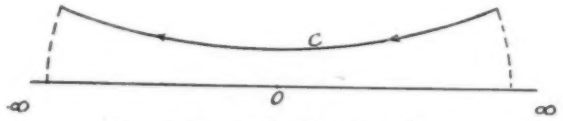

FIG. 1. The path C in the z plane.

that if $A=+J_n(zR)/H_n^{(1)}(zR)$ then a solution which does vanish at $r'=R$ is

$$g(r,\theta;r',\theta',t)=-\frac{1}{4\pi}\sum_{n=-\infty}^{+\infty}\cos n(\theta-\theta')\int_C e^{-\alpha t z^2} \times\frac{H_n^{(1)}(zr)}{H_n^{(1)}(zR)}\{J_n(zr')H_n^{(1)}(zR) -H_n^{(1)}(zr')J_n(zR)\}zdz, \quad r>r'; \qquad (9)$$

when $r<r'$ it is only necessary to interchange r and r'.

In order that Eq. (9) be the correct Green's function it must reduce to zero for all r and r' when $t=0$. That this condition is satisfied by the function (9) is easy to show by means of a suitable contour. The derivative of g is required in Eq. (2) and this is

$$\left.\frac{\partial g}{\partial r'}\right\}_{r'=R}=-\frac{1}{4\pi}\sum_{n=-\infty}^{+\infty}\cos n(\theta-\theta')\int_C e^{-\alpha t z^2} \times\frac{H_n^{(1)}(zr)}{H_n^{(1)}(zR)}\left\{H_n^{(1)}(zR)\frac{d}{d(zR)}J_n(zR) -J_n(zR)\frac{d}{d(zR)}H_n^{(1)}(zR)\right\}z^2dz.$$

Making use of the relation

$$J_n(z)\frac{d}{dz}H_n^{(1)}(z)-H_n^{(1)}(z)\frac{d}{dz}J_n(z)=\frac{2i}{\pi z}$$

the above equation simplifies to

$$\frac{\partial g}{\partial r}=\frac{i}{2\pi^2 R}\sum_{n=-\infty}^{+\infty}\cos n(\theta-\theta')\int_C e^{-at z^2}\frac{H_n^{(1)}(zr)}{H_n^{(1)}(zR)}zdz. \tag{10}$$

Substituting (9) and (10) in (2), the integration with respect to θ' can be carried out at once since neither $f(r')$ nor u_s depend on the angle θ' in our problem. This yields

$$u(r,t)=-\tfrac{1}{2}\int_R^r r'f(r')dr'\int_C e^{-at z^2}\frac{H_0^{(1)}(zr)}{H_0^{(1)}(zR)}\{J_0(zr')H_0^{(1)}(zR)-H_0^{(1)}(zr')J_0(zR)\}zdz$$

$$-\tfrac{1}{2}\int_r^\infty r'f(r')dr'\int_C e^{-at z^2}\frac{H_0^{(1)}(zr')}{H_0^{(1)}(zR)}\{J_0(zr)H_0^{(1)}(zR)-H_0^{(1)}(zr)J_0(zR)\}zdz$$

$$+\frac{ai}{\pi}\int_0^t u_s(\tau)d\tau\int_C e^{-a(t-\tau)z^2}\frac{H_0^{(1)}(zr)}{H_0^{(1)}(zR)}zdz. \tag{11}$$

Since we are interested in the heat flow the derivative of the above expression is required. Making use of the relations

$$\frac{d}{dz}J_0(z)=-J_1(z) \quad \text{and} \quad \frac{d}{dz}H_0^{(1)}(z)=-H_1^{(1)}(z)$$

the derivative is

$$\frac{\partial u}{\partial r}=\tfrac{1}{2}\int_R^r r'f(r')dr'\int_C e^{-at z^2}\frac{H_1^{(1)}(zr)}{H_0^{(1)}(zR)}\{J_0(zr')H_0^{(1)}(zR)-H_0^{(1)}(zr')J_0(zR)\}z^2dz$$

$$+\tfrac{1}{2}\int_r^\infty r'f(r')dr'\int_C e^{-at z^2}\frac{H_0^{(1)}(zr')}{H_0^{(1)}(zR)}\{J_1(zr)H_0^{(1)}(zR)-H_1^{(1)}(zr)J_0(zR)\}z^2dz$$

$$-\frac{ai}{\pi}\int_0^t u_s(\tau)d\tau\int_C e^{-a(t-\tau)z^2}\frac{H_1^{(1)}(zr)}{H_0^{(1)}(zR)}z^2dz. \tag{12}$$

It is not profitable to proceed further with the general formulae without introducing the particular temperature variation at $t=0$, namely $f(r)$, and the way in which the wall temperature changes with the time.

Application to a Special Case

The value of the derivative given by Eq. (12) will be computed for the special case where $f(r)=u_0$ =const. and where the function $u_s(t)$, i.e. the temperature at the wall, is a step function of the time, that is,

$$\begin{aligned} &=u_{s1} && 0<t<t_1\\ u_s=&u_{s2} && t_1<t<t_2\\ &=u_{s3} && t_2<t<t_3\\ &\cdots && \cdots\\ &=u_{sn} && t_{n-1}<t. \end{aligned}$$

Substituting these special functions in Eq. (12) gives

$$\frac{\partial u}{\partial r}=\frac{u_0}{2}\int_R^r r'dr'\int_C e^{-at z^2}\frac{H_1^{(1)}(zr)}{H_0^{(1)}(zR)}\{J_0(zr')H_0^{(1)}(zR)-H_0^{(1)}(zr')J_0(zR)\}z^2dz$$

$$+\frac{u_0}{2}\int_r^\infty r'dr'\int_C e^{-at z^2}\frac{H_0^{(1)}(zr')}{H_0^{(1)}(zR)}\{J_1(zr)H_0^{(1)}(zR)-H_1^{(1)}(zr)J_0(zR)\}z^2dz$$

$$-\frac{ai}{\pi}\left[u_{s1}\int_0^{t_1}d\tau\int_C e^{-a(t-\tau)z^2}\frac{H_1^{(1)}(zr)}{H_0^{(1)}(zR)}z^2dz+u_{s2}\int_{t_1}^{t_2}d\tau\int_C\cdots dz+\cdots+u_{sn}\int_{t_{n-1}}^{t}d\tau\int_C\cdots dz\right].$$

For $r>R$ the integrals with respect to r' and τ can be evaluated thus:

$$\int_R^r zr'J_0(zr')dr'=rJ_1(zr)-RJ_1(zR),$$

$$\int_R^r zr'H_0^{(1)}(zr')dr'=rH_1^{(1)}(zr)-RH_1^{(1)}(zR),$$

$$\int_r^\infty zr'H_0^{(1)}(zr')dr'=-rH_1^{(1)}(zr) \qquad \text{for values of } z \text{ on path } C$$

$$a\int_{t_{k-1}}^{t_k}z^2e^{-a(t-\tau)z^2}d\tau=e^{-a(t-t_k)z^2}-e^{-a(t-t_{k-1})z^2}.$$

Substituting these expressions in the above equation for $\partial u/\partial r$ one obtains:

$$\frac{\partial u}{\partial r}=\frac{u_0}{2}\int_C e^{-at z^2}\frac{H_1^{(1)}(zr)}{H_0^{(1)}(zR)}\{[rJ_1(zr)-RJ_1(2R)]H_0'(zR)-[rH_1^{(1)}(zr)-RH_1^{(1)}(zR)]J_0(zR)\}zdz$$

$$-\frac{u_0}{2}\int_C e^{-at z^2}\frac{rH_1^{(1)}(zr)}{H_0^{(1)}(zR)}\{J_1(zr)H_0^{(1)}(zR)-H_1^{(1)}(zr)J_0(zR)\}zdz$$

$$+\frac{1}{\pi i}\left[u_{s1}\int_C\{e^{-a(t-t_1)z^2}-e^{-at z^2}\}\frac{H_1^{(1)}(zr)}{H_0^{(1)}(zR)}dz+u_{s2}\int_C\{e^{-a(t-t_2)z^2}-e^{-a(t-t_1)z^2}\}\frac{H_1^{(1)}(zr)}{H_0^{(1)}(zR)}dz\right.$$

$$\left.+\cdots+u_{sn}\int_C\{1-e^{-a(t-t_{n-1})z^2}\}\frac{H_1^{(1)}(zr)}{H_0^{(1)}(zR)}dz\right].$$

If it is noted that $\int_C\frac{H_1^{(1)}(zr)}{H_0^{(1)}(zR)}dz=0$ for all $r>R$ and then the limit of the result taken as $r\to R$, the above expression becomes

$$\lim_{r\to R}\frac{\partial u}{\partial r}=-\frac{u_0R}{2}\int_C e^{-at z^2}\frac{H_1^{(1)}(zr)}{H_0^{(1)}(zR)}\{J_1(zR)H_0^{(1)}(zR)-H_1^{(1)}(zR)J_0(zR)\}zdz$$

$$+\frac{1}{\pi i}\left[-u_{s1}\int_C e^{-at z^2}\frac{H_1^{(1)}(zR)}{H_0^{(1)}(zR)}dz+(u_{s1}-u_{s2})\int_C e^{-a(t-t_1)z^2}\frac{H_1^{(1)}(zR)}{H_0^{(1)}(zR)}dz+\cdots\right.$$

$$\left.+(u_{sk}-u_{sk+1})\int_C e^{-a(t-t_k)z^2}\frac{H_1^{(1)}(zR)}{H_0^{(1)}(zR)}dz+\cdots+(u_{sn-1}-u_{sn})\int_C e^{-a(t-t_{n-1})z^2}\frac{H_1^{(1)}(zR)}{H_0^{(1)}(zR)}dz\right].$$

Making use of the relation

$$H_1^{(1)}(zR)J_0(zR)-J_1(zR)H_0^{(1)}(zR)=(-2i/\pi zR)$$

and letting

$$F(t, R)=\frac{1}{\pi i}\int_C e^{-\alpha t z^2}\frac{H_1^{(1)}(zR)}{H_0^{(1)}(zR)}dz$$

the above equation for $(\partial u/\partial r)_{r=R}$ becomes

$$\begin{aligned}(\partial u/\partial r)_{r=R}&=(u_0-u_{s1})F(t, R) && 0<t<t_1\\ &=(u_0-u_{s1})F(t, R)+(u_{s1}-u_{s2})F(t-t_1, R) && t_1<t<t_2\\ &\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots && (13)\\ &=(u_0-u_{s1})F(t, R)+(u_{s1}-u_{s2})F(t-t_1, R)+\cdots+(u_{sn-1}-u_{sn})F(t-t_{n-1}, R)\\ & && t_{n-1}<t<t_n.\end{aligned}$$

For R very large and $u_{s1}=u_{s2}=\cdots=u_{sn}$ the above derivative should approach that for the case of an infinite solid bounded by a plane. To check this, it is only necessary to take the limit as R approaches ∞, i.e.

$$\lim_{R\to\infty}(u_0-u_{s1})F(t, R).$$

For R very large the Hankel functions in $F(t, R)$ may be replaced by their asymptotic expressions

$$H_1^{(1)}(zR)=(2/\pi zR)^{\frac{1}{2}}e^{i(zR-3\pi/4)},$$

$$H_0^{(1)}(zR)=(2/\pi zR)^{\frac{1}{2}}e^{i(zR-\pi/4)},$$

in which case

$$F(t, R)=-\frac{1}{\pi}\int_C e^{-\alpha t z^2}dz.$$

Since the integrand has no poles the integral over C is the negative of that along the real axis or

$$F(t, R)=-\frac{1}{\pi}\int_{-\infty}^{+\infty}e^{-\alpha t x^2}dx=\frac{1}{(\pi\alpha t)^{\frac{1}{2}}}.$$

Thus

$$\lim_{R\to\infty}\frac{\partial u}{\partial r}\Big\}_{r=R}=\frac{u_0-u_{s1}}{(\pi\alpha t)^{\frac{1}{2}}},$$

which is exactly the value of $\partial u/\partial r$ for an infinite solid bounded by a plane.

In order to evaluate $F(t, R)$ it is convenient to let

$$F(t-t_n, R)=(1/R)G(\alpha-\alpha_n),$$

where

$$G(\alpha)=\frac{1}{\pi i}\int_C e^{-\alpha z^2}\frac{H_1^{(1)}(z)}{H_0^{(1)}(z)}dz \qquad (14)$$

and $\alpha=at/R^2$ and $\alpha_n=at_n/R^2$.

The contour integral representing $G(\alpha)$ can be transformed into an integral along the real axis as follows:

$$\begin{aligned}G(\alpha)&=\frac{1}{\pi i}\int_C e^{-\alpha z^2}\frac{H_1^{(1)}(z)}{H_0^{(1)}(z)}dz\\ &=-\frac{1}{\pi i}\int_C e^{-\alpha z^2}\frac{(d/dz)H_0^{(1)}(z)}{H_0^{(1)}(z)}dz\\ &=-\frac{1}{\pi i}\int_C e^{-\alpha z^2}\frac{d}{dz}\log H_0^{(1)}(z)dz.\end{aligned}$$

Integrating by parts,

$$G(\alpha)=-\frac{2\alpha}{\pi i}\int_C ze^{-\alpha z^2}\log H_0^{(1)}(z)dz.$$

Since the integrand has no poles in the positive half-plane and the integral over an indentation at the origin which is a branch point of $H_0^{(1)}(z)$ vanishes as the indentation shrinks to the origin the above integral becomes

$$\begin{aligned}G(\alpha)&=\frac{2\alpha}{\pi i}\int_{-\infty}^{+\infty}xe^{-\alpha x^2}\log H_0^{(1)}(x)dx\\ &=\frac{2\alpha}{\pi i}\int_{-\infty}^{+\infty}xe^{-\alpha x^2}\{\log|H_0^{(1)}(x)|+i\psi(x)\}dx,\end{aligned}$$

where $\psi(x)$ is the argument of $H_0^{(1)}(x)$, i.e.

$\tan \psi(x) = Y_0(x)/J_0(x)$ for $x \geq 0$. Since $|H_0^{(1)}(x)|$ is an even function the above integral reduces to

$$G(\alpha) = \frac{2\alpha}{\pi} \int_{-\infty}^{+\infty} x e^{-\alpha x^2} \psi(x) dx.$$

It may be shown that $\bar{\psi}(x) = \psi(x) + \pi/2$ is an odd function of x. Consequently

$$\int_{-\infty}^{+\infty} x e^{-\alpha x^2} \{\bar{\psi}(x) - \pi/2\} dx = \int_{-\infty}^{+\infty} x e^{-\alpha x^2} \bar{\psi}(x) dx$$

$$= \int_{-\infty}^{+\infty} x e^{-\alpha x^2} \psi(x) dx$$

and the expression for $G(\alpha)$ may finally be written

$$G(\alpha) = \frac{4\alpha}{\pi} \int_0^{\infty} x e^{-\alpha x^2} \bar{\psi}(x) dx. \qquad (15)$$

Approximations for $G(\alpha)$

The integral $G(\alpha)$ cannot be evaluated in terms of simple functions but a rather good approximation of it can be given. This can be done by noting that $x^{\frac{1}{2}}\bar{\psi}(x)$ is very nearly a linear function of x for values of x less than one. The approximation

$$x^{\frac{1}{2}}\psi(x) \leqq Ax, \quad 0 \leq x \leq 1 \qquad (16)$$

will be made for values of $x < 1$ where $A = 1.660$. An idea of the agreement between $x^{\frac{1}{2}}\psi(x)$ and Ax can be had by consulting Table I.

In the interval $x > 1$, $\bar{\psi}(x)$ can be represented very closely by the first three terms of its asymptotic expansion, namely

$$\bar{\psi}(x) \leqq x + \pi/4 - 1/8x. \qquad (17)$$

A reference to Table I will indicate how well the asymptotic expression agrees with $\bar{\psi}(x)$. Substituting the approximate expressions in (15), $G(\alpha)$ becomes

$$\bar{G}(\alpha) = \frac{4\alpha}{\pi} \left\{ A \int_0^1 x^{\frac{1}{2}} e^{-\alpha x^2} dx \right.$$

$$\left. + \int_1^{\infty} (x^2 + \pi/4x - \tfrac{1}{8}) e^{-\alpha x^2} dx. \right\} \qquad (18)$$

Carrying out the integration yields

Table I.

x	$\bar{\psi}(x)$	$x^{\frac{1}{2}}\bar{\psi}(x)$	Ax	$x + \frac{\pi}{4} - \frac{1}{8x}$
0	0	0		
0.02	0.3719	0.0526	0.0332	
.04	.4416	.0883	.0664	
.06	.4923	.1206	.0996	
.08	.5367	.1518	.1328	
.10	.5765	.1823	.1660	
.20	.7415	.3316	.3320	
.30	.8806	.4823	.4980	
.40	1.0079	.6374	.6640	
.50	1.1285	.7980	.8300	
.60	1.2446	.9641	.9960	
.70	1.3577	1.1359	1.1620	
.80	1.4686	1.3135	1.3280	
.90	1.5778	1.4968	1.4940	
1.00	1.6856	1.6856	1.6600	1.6604
1.10	1.7924			1.7717
1.20	1.8984			1.8812
1.30	2.0037			1.9892
1.40	2.1083			2.0960
1.50	2.2125			2.2020
1.60	2.3163			2.3072
1.70	2.4197			2.4119
1.80	2.5227			2.5159
1.90	2.6256			2.6242
2.00	2.7282			2.7229

$$\bar{G}(\alpha) = \frac{8\alpha A}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^n}{(4n+5)n!} + \left(1 + \frac{4}{\pi}\right) \frac{e^{-\alpha}}{2}$$

$$+ \frac{1}{(\pi\alpha)^{\frac{1}{2}}} \left(1 - \frac{\alpha}{4}\right) (1 - \text{Erf } (\alpha)^{\frac{1}{2}}), \qquad (19)$$

where

$$\text{Erf } (\alpha)^{\frac{1}{2}} = \frac{2}{(\pi)^{\frac{1}{2}}} \int_0^{(\alpha)^{\frac{1}{2}}} e^{-s^2} ds.$$

The series is convergent for all values of α but is not particularly convenient for computations for large values of α so that another expression will be given for large α. It may be noted that for very small α the above expression becomes $1/(\pi\alpha)^{\frac{1}{2}}$ which it should do in order to give the flow of heat from an infinite solid bounded by a plane when $R \to \infty$, i.e. $\alpha \to 0$. The above expression has been checked with $G(\alpha)$ for several values of α by integrating Eq. (15) numerically. The agreement is quite good for values of α less than 50. For values of $\alpha > 50$ it is probably best to integrate expression (15) numerically.

For large values of α, i.e. α greater than 3 or 4, it may be shown that

$$\int_0^1 x^{3/2}e^{-\alpha x^2}dx \cong \frac{1}{2\alpha^{5/4}}\left[\Gamma(5/4) - \alpha^{\frac{1}{2}}e^{-\alpha}\left(1+\frac{1}{4\alpha}-\frac{3}{16\alpha^2}+\cdots\right)\right]$$

and

$$(1-\text{Erf }(\alpha)^{\frac{1}{2}}) = \frac{e^{-\alpha}}{(\pi\alpha)^{\frac{1}{2}}}\left[1-\frac{1}{2\alpha}+\frac{3}{(2\alpha)^2}\cdots\right].$$

Substituting these expressions in Eq. (18) one finds that

$$\bar{G}(\alpha) = \frac{2}{\pi}\left[\frac{A\Gamma(5/4)}{\alpha^{\frac{1}{2}}} - e^{-\alpha}\left\{\left(A-\frac{\pi}{4}-\frac{7}{8}\right) + \frac{1}{\alpha}\left(\frac{4A-9}{16}\right) - \frac{1}{\alpha^2}\left(\frac{11-6A}{32}\right)+\cdots\right\}\right], \quad (20)$$

where $\Gamma(5/4)$ is the gamma-function of 5/4 or $\Gamma(5/4)=0.9064$.

Total Heat Flow

The total amount of heat which has flowed through the cylinder wall during the time interval $t=0$ to $t=t$ is given by

$$Q = k\int_0^t \left.\frac{\partial u}{\partial r}\right\}_{r=R} dt,$$

where k is the heat conductivity. From Eq. (13) this becomes

$$\begin{aligned}
Q &= k(u_0-u_{s1})\int_0^t F(t,R)dt \qquad 0<t<t_1 \\
&= k\left[(u_0-u_{s1})\int_0^t F(t,R)dt + (u_{s1}-u_{s2})\int_{t_1}^t F(t-t_1,R)dt\right] \qquad t_1<t<t_2 \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots \qquad (21) \\
&= k\left[(u_0-u_{s1})\int_0^t F(t,R)dt + (u_{s1}-u_{s2})\int_{t_1}^t F(t-t_1,R)dt + \cdots \right. \\
&\quad \left. + (u_{sn-1}-u_{sn})\int_{t_{n-1}}^t F(t-t_{n-1},R)dt\right] \\
&\qquad\qquad t_{n-1}<t<t_n.
\end{aligned}$$

Consequently an integral of the type

$$\int_{t_m}^t F(t-t_m, R)dt$$

must be evaluated. From Eq. (14) it is seen that

$$\int_{t_m}^t F(t-t_m,R)dt = \frac{1}{R}\int_{t_m}^t G(\alpha-\alpha_m)dt = \frac{R}{a}\int_0^{\alpha-\alpha_m} G(\alpha) = \frac{R}{a}K(\alpha-\alpha_m). \quad (22)$$

With the approximate expression (19) the integrations indicated above can be carried out giving

$$K(\alpha-\alpha_m) = \frac{2}{\pi}\left[4A(\alpha-\alpha_m)^2\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha-\alpha_m)^n}{(4n+5)(n+2)n!} + \left(\frac{\pi}{4}+\frac{23}{12}\right)(1-e^{-(\alpha-\alpha_m)}) + \frac{(\alpha-\alpha_m)}{12}e^{-(\alpha-\alpha_m)} + (\pi(\alpha-\alpha_m))^{\frac{1}{2}}\left(1-\frac{\alpha-\alpha_m}{12}\right)(1-\text{Erf }(\alpha-\alpha_m)^{\frac{1}{2}})\right]. \quad (23)$$

Although the series in the above expression converges for all values of $(\alpha-\alpha_m)$ it is not convenient to use for large values of $(\alpha-\alpha_m)$ so that an asymptotic series is desirable for large values of $(\alpha-\alpha_m)$. This may be obtained most conveniently by going back to the first integral in Eq. (18) from which the above series originated. It is then seen that a value of the integral

$$I = \int_0^{\alpha-\alpha_m} \alpha d\alpha \int_0^1 x^{\frac{1}{2}}e^{-\alpha x^2}dx$$

is required for large values of $(\alpha-\alpha_m)$. By means of an obvious transformation I may be written

$$I = \int_0^{\beta}\alpha^{-\frac{1}{2}}d\alpha\int_0^{\alpha} s^{\frac{1}{2}}e^{-s}ds, \quad (24)$$

where $\beta=\alpha-\alpha_m$. The integral

$$\int_0^{\alpha} s^{\frac{1}{2}}e^{-s}ds$$

is just the integral representation of the incomplete gamma-function which with the help

of the confluent hypergeometric function[4] may be written as follows:

$$\int_0^{\alpha} s^{\frac{1}{4}} e^{-s} ds = \Gamma(5/4) - \alpha^{\frac{1}{4}} e^{-\alpha} \int_0^{\infty} (1+\xi/\alpha)^{\frac{1}{4}} e^{-\xi} d\xi.$$

Substituting this in Eq. (24) and inverting the order of integration and integrating with respect to α one finally obtains

$$I = \frac{4}{3}\Gamma(5/4)\beta^{3/4} - \frac{4}{3} + \frac{e^{-\beta}}{\beta}\left[\int_0^{\infty} (1+y/\beta)^{-3/4} e^{-y} dy + \int_0^{\infty} (1+y/\beta)^{-7/4} e^{-y} dy\right].$$

Expanding the integrands in powers of y/β and integrating term by term, the required asymptotic series is

$$I \doteqdot \frac{4}{3}\Gamma(5/4)\beta^{3/4} - \frac{4}{3} + \frac{e^{-\beta}}{\beta}\left[2 - \left(\frac{3}{4}+\frac{7}{4}\right)\frac{1}{\beta} + \frac{7}{4}\left(\frac{3}{4}+\frac{11}{4}\right)\frac{1}{\beta^2} - \frac{7}{4}\cdot\frac{11}{4}\left(\frac{3}{4}+\frac{15}{4}\right)\frac{1}{\beta^3} + \cdots\right].$$

Using this expression the complete expression for $K(\alpha-\alpha_m)$ is

$$K(\alpha-\alpha_m) \doteqdot \frac{2}{\pi}\left[\frac{8A}{3}\Gamma(5/4)(\alpha-\alpha_m)^{3/4} - \frac{8A}{3} + \frac{2Ae^{-(\alpha-\alpha_m)}}{(\alpha-\alpha_m)}\left\{2 - \left(\frac{3}{4}+\frac{7}{4}\right)\frac{1}{\beta} + \frac{7}{4}\left(\frac{3}{4}+\frac{11}{4}\right)\frac{1}{\beta^2} - \frac{7}{4}\cdot\frac{11}{4}\left(\frac{3}{4}+\frac{15}{4}\right)\frac{1}{\beta^3} + \cdots\right\}\right. \quad (25)$$

$$\left. + \left(\frac{\pi}{4}+\frac{23}{12}\right)(1-e^{-(\alpha-\alpha_m)}) + \frac{\alpha-\alpha_m}{12}e^{-(\alpha-\alpha_m)} + (\pi(\alpha-\alpha_m))^{\frac{1}{2}}\left(1-\frac{\alpha-\alpha_m}{12}\right)(1-\mathrm{Erf}\,(\alpha-\alpha_m)^{\frac{1}{2}})\right]$$

$$\beta = (\alpha-\alpha_m) > 3.$$

For large values of α, i.e. $4<\alpha<50$, the above expression is very convenient and quite accurate since many of the terms yield only small contributions.

In terms of the dimensionless quantity $K(\alpha-\alpha_m)$ the expression (21) for the total heat which has flowed up to the time t per unit area of tunnel wall is

$$Q = \frac{kR}{a}(u_0-u_{s1})K(\alpha-0) \qquad 0<t<t_1$$

$$= \frac{kR}{a}[(u_0-u_{s1})K(\alpha-0)+(u_{s1}-u_{s2})K(\alpha-\alpha_1)] \qquad t_1<t<t_2$$

$$= \cdots\cdots\cdots\cdots$$

$$= \frac{kR}{a}[(u_0-u_{s1})K(\alpha-0)+(u_{s1}-u_{s2})K(\alpha-\alpha_1) + \cdots + (u_{sn-1}-u_{sn})K(\alpha-\alpha_{n-1})] \qquad t_{n-1}<t<t_n.$$

It does not seem practical to carry the analysis any further for the special case where the tunnel wall temperature is a step function of the time. The above formula is quite useful in all those cases for which the wall temperature can be approximated by a step function.

[4] Whittaker and Watson, p. 341.

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CURRENT LITERATURE OF PHYSICS

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Annales de physique

SER. 11, VOL. 7, APRIL, 1937.

Ferromagnetic Curie Points and Absolute Saturation of Nickel Alloys. pp. 459–527. ***V. Marian***

Near Infrared Spectra of Organic Substances in Liquid and Vapor States. pp. 528–598. ***A. Naherniae***

Annalen der Physik

SER. 5, VOL. 28, NO. 7, MARCH 18, 1937.

Absorption Spectra of Praseodymium Salts and Their Zeeman Effect. pp. 569–602. ***A. Merz***

Experimental Determination of Mean Free Path of Electrons in Lead and Cadmium. pp. 603–631. ***L. Riedel***

Study of Relation of Dust Patterns Due to Transverse Vibrations of Square and Rectangular Plates. pp. 632–648. ***B. Pavlik***

Anomalous Dispersion of Excited Argon. pp. 649–666. ***K. Schön***

Hyperfine Structure of Sodium D Lines. pp. 667–672. ***K. Meissner and K. Luft***

SER. 5, VOL. 28, NO. 8, MARCH 20, 1937.

Absorption and Fluorescence Spectra of Ions of the Rare Earths in Solids, Particularly in the Infrared. pp. 673–700. ***H. Gobrecht***

Modification of Ferromagnetic Properties of Cubic Single Crystals by Mechanical Tension. pp. 701–720. ***H. Schlechtweg***

Theory of Charged Elementary Particles. pp. 721–760. ***H. Hönl***

SER. 5, VOL. 29, NO. 1, APRIL 8, 1937.

Transmutation of Boron by Slow Neutrons. pp. 1–10. ***E. Fünfer***

Theory of Atomic Photoelectric Effect. pp. 11–24. ***F. Renner***

Thermoelectric Power of Bismuth Crystals with Transverse and with Longitudinal Magnetic Field. pp. 25–32. ***E. Grüneisen and J. Gielessen***

Ion Collisions with Strong Resonance. pp. 33–47. ***F. Wolf***

Structure of Dipole Liquids. pp. 48–60. ***J. Malsch***

Dielectric Constants and Dipole Losses of Glasses at High Frequencies. pp. 63–80. ***W. Hackel***

Blocking Layer Photo-Cells Studied with Soft X-Rays. pp. 81–96. ***H. Felsinger***

Canadian Journal of Research

VOL. 15, NO. 4, APRIL, 1937.

Thermal Stress in Long Cylindrical Shells Due to Temperature Variation Round the Circumference, and Through the Wall. pp. 49–58. ***J. N. Goodier***

Specific Heat of Copper from 30° to 200°K. pp. 59–66. ***S. M. Dockerty***

Comptes rendus

VOL. 204, NO. 12, MARCH 22, 1937.

Theoretical Consideration of Photodichroism. pp. 973–975. ***S. Nikitine***

Regularities in Nuclear Levels of Radioactive Atoms. pp. 975–977. ***S. Rosenblum and M. Guillot***

VOL. 204, NO. 15, APRIL 12, 1937.

Dirac Mechanics and the Last Multiplier in the Jacobi Sense. pp. 1104–1106. ***R. Dugas***

Spectrographic Study of Conductivity Electrons of Magnesium and Silicon. pp. 1108–1110. ***J. Farineau***

Study of Certain Borates and Oxides in the Far Infrared. pp. 1111–1112. ***M. Parodi***

Calculation of Vibration Frequencies of N_2O_4. pp. 1112–1114. ***J. Duchesne***

Apparatus for Obtaining Intense Diffraction Patterns from Powdered Crystals with Monochromatic Radiation. pp. 1115–1116. ***A. Guinier***

Theory of β-Radioactivity. pp. 1117–1119. ***A. Mercier***

Journale de physique

SER. 7, VOL. 8, MARCH, 1937.

Space-Time Description of Quantum Phenomena. pp. 81–87. ***B. Kwal***

Elementary Theory of Interference Apparatus. pp. 88–92. ***A. Bogros***

Modifications and Transformations of Metals by Heating in Vacuum and in Various Gases. II. pp. 93–98. ***J. Trillat and S. Oketani***

Theory of Latent Energy of Solid Solutions. pp. 99–102. ***J. Cichocki***

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VOL. 8, NO. 5, MAY, 1937.

Editorial. p. 297.

National Bureau of Standards. pp. 298–306. *Lyman J. Briggs*

Uses and Limitations of X-Ray Diffraction Methods. pp. 307–312. *John T. Norton*

A Lightning Flash and Its Component Strokes. pp. 313–318. *John G. Albright*

The Liquid State. pp. 319–327. *Karl E. Herzfeld*

Flow Phenomena in Heavily Stressed Metals. pp. 328–336. *P. W. Bridgman*

Edward Bausch—A Biography pp. 337–341. *Everett W. Melson*

Minute Nonferritic Particles in Steel. pp. 342–350. *A. B. Kinzel*

Contributed Original Research:

Study of Magnetic Losses at Low Flux Densities in 35 Permalloy Sheet. pp. 351–358. *W. B. Ellwood and V. E. Legg*

Heat Effects in Capillary Flow at High Rates of Shear. pp. 359–363. *M. D. Hersey and J. C. Zimmer*

Note on the Filtering Action of Porous Media. pp. 363–367. *W. N. Arnquist*

Effects of Pressure and Temperature on the Viscosity of Lubricating Oils. pp. 367–372. *R. B. Dow*

Journal of the Optical Society

VOL. 27, NO. 5, MAY, 1937.

Graphical Exposition of the Michelson-Morley Experiment. pp. 177–180. *Herbert E. Ives*

Optical Constants of Sodium. pp. 181–185. *H. E. Ives and H. B. Briggs*

Brightness of Cathodo-Luminescence at Low Current Densities and Low Voltages. pp. 186–192. *Thomas B. Brown*

High Speed High Precision Microphotometer. pp. 193–197. *H. B. Vincent and R. A. Sawyer*

Journal of Research of the National Bureau of Standards

VOL. 18, NO. 3, MARCH, 1937.

Anodizing of Aluminum Alloys in Chromic Acid Solutions of Different Concentrations. pp. 251–257. *Robert W. Buzzard*

Cooperative Study of Methods for the Determination of Oxygen in Steel. pp. 259–293. *John G. Thompson et al.*

New Phenomenon in Superconducting Transition of Tantalum and of Tin. pp. 295–314. *Francis B. Silshee et al.*

VOL. 18, NO. 4, APRIL, 1937.

Calorimetric Determination of the Thermodynamic Properties of Saturated Water in Both the Liquid and Gaseous States from 100° to 374°C. pp. 389–447. *N. S. Osborne et al.*

Precision Camera for Testing Lenses. pp. 449–460. *I. C. Gardner and F. A. Case*

Journal of Scientific Instruments

VOL. 14, NO. 5, MAY, 1937.

Spectroscopy in Industry. pp. 153–160. *A. Fowler*

Application of the Gaseous Discharge Tube to Problems of Voltage Regulation. pp. 161–164. *C. Morton*

Modified Potentiometer for Measuring Very Small Resistances. p. 165. *P. Kapitza and C. J. Milner*

Method of Absorption Spectrography for the Detection of the Detailed Structure of Absorption Spectra. pp. 166–172. *E. R. Holiday*

New Method for Recording Minute Changes of Capacitance. pp. 173–176. *W. D. Oliphant*

Laboratory and Workshop Notes:

Adjustable Speciman Holder for a Debye-Scherrer Camera. pp. 177. *F. W. Jones and H. Lipson*

Receiver Changer for Small Scale Vacuum Distillations. p. 178. *W. V. Thorpe and R. T. Williams*

New Instruments and Tools:

Laboratory Electric Hot Plates. p. 180.

Illumination Device for Microscopes. p. 181.

A.C. Recording Instrument. p. 182.

Recording Microphotometer. p. 183.

Nature

VOL. 139, NO. 3519, APRIL 10, 1937.

Letters to the Editor:

Effects of Traces of Metallic Ions on Monolayers. p. 625. *J. S. Mitchell et al.*

Adsorption on Foam. p. 629. *Franz Schütz*

Absorption Spectrum of the Carboxyl Group in the Vacuum Ultra-Violet. p. 630. *W. M. Evans and W. C. Price*

Sensitive Adaptation of the Spoon Gauge. p. 631. *A. Lewis and D. W. G. Style*

Molecular, Nematic and Crystal States of 1 : 1′-Diethyl-ψ-Cyanine Chloride. p. 631. *Edwin E. Jelley*

Absorption of Strain Energy in Metals. p. 633. *N. A. de Bruyne*

Electrical Stimulation of the Cochlea. p. 633. *Ernst Bárány*

Diurnal Variation of Cosmic Ray Shower. p. 633. *M. Forró*

Potential Constants of Tetrachlorethylene. p. 634. *Jules Duchesne*

VOL. 139, NO. 3520, APRIL 17, 1937.

Letters to the Editor:

Equilibrium and Kinetics in the Gaseous State and in Solution. pp. 669. *G. A. Benford et al.*

Origin of the Cochlear Effect. p. 670. *O. Gatty and A. F. Rawdon-Smith*

Phase Velocity of Electro-Magnetic Waves Along the Ground. p. 671. *W. Ross and E. C. Slow*

Cosmic Ray Showers. p. 671. *Y. Watase*

Action of Iron Oxide on the Emission Spectrum of Nitrogen-Mercury Systems Excited by α-Rays. p. 672. *André Luyckx*

Eddy-Current Resistance in Fluids Due to Rotation. p. 673. *Kerr Grant*

Magnetic and Electrical Dimensions. p. 676. *J. Henderson and W. Cramp*

Solar and Magnetic Conditions Associated with Recent Auroras. p. 680. *J. P. Rowland, S.J.*

VOL. 139, NO. 3521, APRIL 24, 1937.

Letters to the Editor:

Atmospheric Disturbances Due to Thundercloud Discharges. p. 711. *F. W. Chapman*

Chemical Detection of Helium Formed in Beryllium by Gamma-Rays. p. 712. *F. A. Paneth and E. Glückauf*

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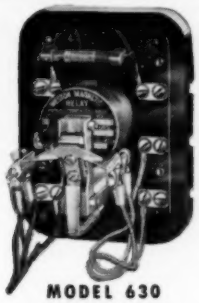

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Induced Radioactivity with Neutrons from Slow Deuterons. p. 716. ***E. Bertl et al.***

Photo-Electric Method for the Measurement of the Optical Constants of Metals. pp. 716. ***J. Bor***

Perception of Weight and Elasticity. p. 719. ***D. Katz and W. Stephenson***

VOL. 139, NO. 3522, MAY 1, 1937.

Letters to the Editor:

Lead Extrusion Phenomenon. p. 755. ***P. Dunsheath***

Method of Measuring the Magnetic Moment of Free Neutrons. p. 756. ***O. R. Frisch et al.***

Radioactive Neon. p. 757. ***T. Bjerge***

Cosmological Constants. p. 757. ***S. Chandrasekhar***

Sources of Inaccuracies in Spectographic Research. p. 758. ***E. J. B. Willey***

Naturwissenschaften

VOL. 25, NO. 15, APRIL 9, 1937.

Deuteron-Deuteron Reaction. pp. 231–232. ***H. Kallmann and E. Kuhn***

Structure of Thin Metallic Films Condensed at Low Temperatures. pp. 232–233. ***G. Hass***

Evidence for Slow Neutrons in the Atmosphere. p. 235. ***E. Fünfer***

Philosophical Magazine

VOL. 23, NO. 155, APRIL, 1937.

Effect of Thunderstorms and Magnetic Storms on the Ionization of the Kennelly-Heaviside Layer. pp. 513–528. ***J. N. Bhar and P. Syam***

Graphical Estimation of the Signal Handling Capacity of Screen-Grid Valves. pp. 529–534. ***Robert W. Sloane***

Corona Discharge Between Points in Hydrogen. pp. 534–545. ***Robert W. Sloane***

Lattice Spacings of Iron-Nickel Alloys. pp. 545–557. ***A. J. Bradley et al.***

On the Fluorescence of Cyclohexane. pp. 558–562. ***S. M. Mitra***

Frequency of Transverse Vibration of a Loaded Fixed-Free Bar.—III. The Effect of the Rotatory Inertia of the Bar. pp. 563–572. ***R. M. Davies***

On the Motion of an Extensible Membrane in a Given Curved Surface. pp. 573–580. ***C. E. Weatherburn***

Deflexion of Unsupported Beams Under Variable Loads. pp. 580–590. ***R. A. Fairthorne***

Vibrations of a Face-Centered Cubic Lattice. pp. 590–597. ***C. Strachan***

Radial and Spiral Flow of a Compressible Fluid. pp. 597–606. ***A. M. Binnie and S. G. Hooker***

General Proof of Saint-Venant's Principles. pp. 607–609. ***B. Dasannacharya and T. S. Krishna Moorthy***

Effect of Heat Treatment on the Tribo-Electric Properties of Quartz and of Metals. pp. 620–628. ***P. A. Mainstone***

Geodesic Postulate in General Relativity. pp. 628–632. ***V. V. Narlikar and Jaipat Singh***

SUPPLEMENTARY NUMBER, VOL. 23, NO. 156, APRIL, 1937.

Ignition of Gaseous Mixtures by Hot Particles. pp. 633–657. ***Robert S. Silver***

On the Production of Auroral and Night-Sky Light. pp. 657–665. ***S. Chapman***

On Acoustic Impedance Bridge. pp. 665–681. ***N. W. Robinson***

Aerodynamic Characteristics of a Cylinder Having a Heated Boundary Layer. pp. 681–695. ***E. G. Richardson***

On the Construction and Sensitivities of Short Period Platinum Thermometers. pp. 695–701. ***J. J. Manley***

Tribo-Electric Properties of a Quartz-Nickel Interface Between 120°C and −78°C. pp. 702–708. ***P. A. Mainstone***

Effect of Trochoidal Interference on the Operation of Involute Gears. pp. 708–717. ***W. A. Tuplin***

Effect of Air Velocity on the Spark Between Point-Plate and Two-Point Electrode Systems in Atmospheric Air. pp. 718–726. ***E. L. E. Wheatcroft et al.***

The Plane Jet. pp. 727–731. ***W. G. Bickley***

Methods for the Calculation of Lift Distributions of Aerofoils. pp. 731–744. ***R. C. J. Howland***

Symmetrical Flexure of an Angle-Iron. pp. 745–757. ***B. R. Seth***

Notes on the Two-Dimensional Potential Theory.—II. Hydrodynamical Problems on the Motion of Cylinders. pp. 757–762. ***Rosa M. Morris***

Contour Integral Expressions for Bessel Functions. pp. 762–774. ***N. W. McLachlan and H. L. Meyers***

Motions of Electrons in a Gas in the Presence of Variable Electric Fields and a Constant Magnetic Field. pp. 774–791. ***V. A. Bailey***

Magneto-Optical Dispersion of Organic Liquids in the Ultraviolet Region of the Spectrum.—Part X. The Magneto-Optical Dispersion of Iso-Amyl Acetate, Methyl Iso-Valerate, and Acetone. pp. 791–806. ***Beryl P. M. Walters and E. J. Evans***

Note on Laguerre Polynomials. pp. 807–811. ***W. T. Howell***

Theory of Electrosmotic Circulation in Closed Vessels. pp. 811–823. ***Paul White***

Improved Method of Measuring Electrophoresis by the Ultramicroscope. pp. 824–827. ***T. B. Laue and Paul White***

Physica

VOL. 4, NO. 4, APRIL, 1937.

Magnetic Powder Experiments on Rolled Nickel Iron. II. pp. 257–266. ***J. Snoek and M. Louwerse***

Secondary Electron Emission of Soot in Valves with Oxide Cathode. pp. 267–275. ***H. Bruining et al.***

Thermoelectric Force of a Superconductor Against the Same Metal in Non-Superconducting State. pp. 276–278. ***J. Kok***

Gradient in Very High Pressure Mercury Discharge. pp. 279–284. ***W. Elenbaas***

Optical Investigation of the Latent Image and the Print-Out Effect of Photographic Emulsions. pp. 285–296. ***A. v. Kreveld and H. Jurriens***

Method for Determination of Minute Absorptions of Light. pp. 297–304. ***A. v. Kreveld and H. Jurriens***

Determination of Vapor Pressures of Liquid Nitrogen Below 1 Atm. and of Solid Nitrogen β. pp. 305–310. ***W. Keesom and A. Bijl***

Piezoelectric Constant of Quartz. pp. 311–315. ***J. Clay and J. Karper***

Relative Probability of Excitation of the three L-levels of Tungsten as a Function of the Velocity of Cathode Rays. pp. 317–324. ***W. Huizinga***

Disturbance of Superconductive State by a Magnetic Field. pp. 325–328. ***W. de Haas and A. Engelkes***

Properties of Condensed Phases of Helium and Hydrogen. pp. 329–344. ***A. Bijl***

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Physical Review

VOL. 51, NO. 9, MAY 1, 1937.

Gamma-Rays from Light Elements Due to Proton Bombardment. pp. 691–698. ***R. G. Herb et al.***

Construction and Operation of a Cyclotron to Produce One Million Volt Deuterons. pp. 699–705. ***P. G. Kruger and G. K. Green***

Collisions of Neutron and Proton. II. pp. 706–710. ***P. M. Morse et al.***

Distribution in Time of Counts Due to a Constant Source and Its Daugher in Equilibrium. pp. 710–712. ***L. Devol and A. Ruark***

Shape of the Beta-Ray Distribution Curve of Radium E at High Energies. pp. 713–717. ***L. M. Langer and M. D. Whitaker***

Effect of Galactic Rotation Upon the Intensity of Cosmic Radiation. pp. 718–720. ***W. F. G. Swann***

Auger Effect in Germanium. pp. 720–725. ***Michael Ference, Jr.***

Spectra of Phosphorus. Part II: The Spectra of Doubly, Triply and Quadruply Ionized Phosphorus (P III, P IV, P V). Additions and Corrections to P II. pp. 726–735. ***Howard A. Robinson***

Correction of Continuous Spectra for the Finite Resolution of the Spectrometer. pp. 735–738. ***Carl Eckart***

Pressure Effect in Bands of Several Dipole Molecules. pp. 739–744. ***S. D. Cornell***

Drift of Ions and Electrons in a Magnetic Field. pp. 744–747. ***Lewi Tonks***

Long Range Interactions Between Dipole Molecules. pp. 748–753. ***H. Margenau and D. T. Warren***

Vapor Pressure of Caesium by the Positive Ion Method. pp. 753–760. ***J. B. Taylor and I. Langmuir***

Electronic Energy Bands in Metallic Calcium. pp. 761–764. ***M. F. Manning and H. M. Krutter***

On the Choice of the Action Function in the New Field Theory. pp. 765–773. ***B. Hoffmann and L. Infeld***

Letters to the Editor:

Magnetic Anisotropy of Tetracovalent Co^{++} Ions in Crystals. p. 774. ***K. S. Krishnan and A. Mookherji***

Width of Iodine Resonance Neutron Band. p. 774. ***S. Ruben and W. F. Libby***

Scattering of Neutrons by Ortho- and Parahydrogen. p. 775. ***J. Schwinger and E. Teller***

Proton-Induced Radioactivity in Heavy Nuclei. p. 775. ***S. W. Barnes et al.***

Photoelectric Investigation of the "Allison Magneto-Optic Effect." p. 776. ***George C. Comstock***

On the Saturation Property of Nuclear Forces. p. 777. ***Eugene Feenberg***

Paramagnetism at Radiofrequencies. p. 778. ***C. J. Gorter***

Approximately Relativistic Equations for Nuclear Particles. Addendum. Proof of Approximate Invariance. p. 778. ***G. Breit***

Note on the Reaction of Deuterium on Deuterium. p. 779. ***M. H. Johnson, Jr.***

Erratum: Heights of Reflection of Radio Waves in the Ionosphere. p. 779. ***F. H. Murray***

Isotope Shift of Li I. p. 780. ***Dominic d'Eustachio***

Remanence in Single Crystals. p. 780. ***K. J. Sixtus***

Remarks on the Paper by E. B. Baker and H. A. Boltz "Thermionic Emission into Dielectric Liquids." p. 781. ***Karl Hans Reiss***

Superconductivity. p. 781. ***Kurt Mendelssohn***

Production of X-Rays by High Speed Argon Ions. p. 781. ***M. Tanaka and I. Nonakap***

Physikalische Zeitschrift

VOL. 38, NO. 8, APRIL 15, 1937.

Structure of Electrons and Supraconductivity. pp. 269–277. ***J. Stark***

Explanation of Magnetic Induction in Supraconductors. pp. 277–283. ***J. Stark and K. Steiner***

Dipole Moment of Chlorbenzol and Its Behavior in 34 Solvents. pp. 283–292. ***H. Müller***

Volume Rectifier Action in Illuminated Cuprous Oxide Crystals. pp. 292–298. ***G. Groetzinger and J. Lichtschein***

Molecular Transformations of Higher Order. pp. 298–303. ***N. Moerman and G. Muller***

Physikalische Zeitschrift der Sowjetunion

VOL. 11, NO. 1, 1937.

Critique of Neutrino Theory of Light. p. 1–8. ***V. Fock***

Comment on Fermi Theory of Positron Decay. pp. 9–17. ***C. Möller***

Sound Absorption in Solids. pp. 18–25. ***L. Landau and G. Rumer***

Theory of Phase Transformations. pp. 26–47. ***L. Landau***

Proton-Neutron Transformation under the Action of Gamma-Rays. pp. 48–54. ***G. Rumer***

Specific Heat of Anhydrous $MnCl_2$. pp. 55–59. ***O. Trapeznikowa and G. Miljutin***

Enthalpy Diagram of Nitrogen from 60 to 200 Atmospheres. pp. 60–68. ***I. Gusak***

Load Measurement of Magnetron Generators for Decimeter Waves. pp. 69–77. ***S. Braude and A. Slutzkin***

Angular Distribution of Photo-Neutrons from Beryllium. pp. 78–82. ***T. Goloborodko and L. Rosenkewitsch***

Theory of Fine Structure in X-Ray Absorption Spectra of Molecular Gases. pp. 83–94. ***B. Bogdanovich***

Theory of Corona Discharge. pp. 95–117. ***N. Kapzov***

Data on Connection Between Secondary Electron Emission and Photosensitivity. pp. 118–121. ***A. Dobroljubski***

Properties of Blocking Layer of Photo-Cells. pp. 122–127. ***W. Jouse***

VOL. 11, NO. 2, 1937.

Theory of Supraconductivity. pp. 129–140. ***L. Landau***

Electron Gas in a Magnetic Field. pp. 141–156. ***E. Lifschitz***

Method for Investigation of X-Ray Interference in Liquids. pp. 157–167. ***D. Dzelepow***

Effect of Previous Illumination and Temperature on Absorption of Light by Cuprous Oxide. pp. 168–181.

Energy of Electromagnetic Field in the Dynamic Theory of X-Ray Interference and Quantum Mechanical Considerations. pp. 182–193. ***F. Galperin***

Magnetic Susceptibility of Vapors of Organic Compounds. pp. 194–203. ***J. Shur***

Dependence of β-Spectra of Radioactive Elements on the Atomic Number. pp. 204–224. ***A. Alichanian et al.***

Scattering of Foot Neutrons by Protons. pp. 225–238. ***N. Omitrieff***

Proceedings of the Cambridge Philosophical Society

VOL. 33, PART 2, APRIL, 1937.

Effect of Configuration Interaction on the Low Terms of the Spectra of Oxygen. pp. 240–249. ***D. R. Hartree and B. Swirles***

Note on Molecular Orbitals. pp. 250–252. ***A. F. Devonshire***

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Comparison of Wave Functions for the Normal Helium Atom. pp. 253–259. ***T. D. H. Baber and H. R. Hassé***

Superconducting Cylinders. pp. 260–276. ***D. Shoenberg***

Generalized Fermi Interaction. pp. 277–292. ***F. Hoyle***

Proceedings of the Royal Society

VOL. 159, NO. 898, APRIL 1, 1937.

Conformal Transformation with the Aid of an Electric Tank. pp. 315–346. ***K. N. E. Bradfield et al.***

On the Surface Cracks in Glasses. pp. 346–354. ***E. N. daC. Andrade and L. C. Tsien***

On the New Field Theory. II.—Quantum Theory of Field and Charges. pp. 355–382. ***M. H. L. Pryce***

An Investigation of the Production of Positive and Negative Electron Pairs in a Cloud Chamber. pp. 383–394. ***L. Simons and K. Zuber***

Diamagnetic Susceptibilities of Salts Forming Ions with Inert Gas Configurations. III—The Alkaline Earth Halides and General Discussion. pp. 395–409. ***F. E. Hoare and G. W. Brindley***

Near Infrared Absorption Spectrum of Heavy Water. pp. 410–415. ***L. Kellner***

On the Vibrational Spectrum of a Three-Dimensional Lattice. pp. 416–431. ***M. Blackman***

Passage of Fast Electrons and the Theory of Cosmic Showers. pp. 432–458. ***H. J. Bhabha and W. Heitler***

Conical Shock Wave Formed by a Cone Moving at a High Speed. pp. 459–472. ***J. W. Maccall***

VOL. 159, NO. 899, APRIL 15, 1937.

Similarity Theory of Turbulence, and Flow Between Parallel Planes and Through Pipes. pp. 473–496. ***S. Goldstein***

Flow in Pipes and Between Parallel Planes. pp. 496–506. ***G. I. Taylor***

Standard Source of Sound and the Measurement of Minimum Audibility. pp. 507–526. ***E. N. daC. Andrade and R. C. Parker***

Kinematics, Dynamics and the Scale of Time. III. pp. 526–547. ***E. A. Milne***

General Motion of a Spinning Uniformly and Rigidly Electrified Sphere. III. pp. 548–570. ***G. A. Schott***

Uniform Circular Motion with Invariable Normal Spin of a Rigidly and Uniformly Electrified Sphere. IV. pp. 570–591. ***G. A. Schott***

Flow Under Gravity of an Incompressible and Inviscid Fluid Through a Constriction in a Horizontal Channel. pp. 592–609. ***A. M. Binnie and S. G. Hooker***

Review of Scientific Instruments

VOL. 8, NO. 5, MAY, 1937.

Physics Forum:

Double Ionization of Atoms. pp. 137–139. ***F. K. Richtmyer***

Third Washington Conference on Theoretical Physics. pp. 141–142. ***G. Breit***

Symposium on Physics in the Textile Industry. pp. 143–144. ***H. A. B.***

Contributed Articles:

Tube for the Production of Monochromatic and Fluorescent X-Rays. pp. 145–147. ***W. C. Pierce et al.***

Vacuum Spectrograph for X-Rays. pp. 147–150. ***C. M. Olson and W. C. Pierce***

Simple Method for Measuring the Coefficient of Non-viscous Friction of Thin Lubricating Layers. pp. 151–158. ***S. Kyropoulos and E. I. Shobert***

High Frequency Filament Supply for Ion Sources. pp. 158–159. ***J. R. Dunning and H. L. Anderson***

Inverted Air-Driven Ultracentrifuge. pp. 160–161. ***J. W. Beams and F. W. Linke***

Thermionic Control of an Ionization Gauge. pp 160–164. ***L. N. Ridenour and C. W. Lampson***

Simple Electrostatic Voltmeter for High Voltages. pp. 165–169. ***C. W. Lampson***

Laboratory and Shop Notes:

Simplified Method of Placing Extra Necks on Round Bottom Flasks. p. 170. ***Walter O. Osborn et al.***

Quartz-to-Pyrex Joint. p. 170. ***A. H. Weber and C. B. Bazzoni***

Modified Smythe Vacuum Leak. p. 171. ***C. B. Bazzoni***

Sensitive Beta-Ray Counter. p. 172. ***J. Halpern and O. C. Simpson***

New Type of Electromagnet. p. 172. ***Harvey Allison***

Zeitschrift für Physik

VOL. 105, NO. 1–2, APRIL 6, 1937.

Temperature of Carbon Arc from CN Bands. pp. 1–15. ***W. Lochte-Holtgreven and H. Maecker***

Magnetic Moment of Scandium Nucleus. pp. 16–20. ***H. Kopfermann and H. Wittke***

Allotropic Transformations. pp. 21–35. ***U. Dehlinger***

Vibration Terms and Rotation Constants of the $a'\,^3\Sigma^+$ State of Carbon Monoxide. pp. 36–44. ***R. Schmid and L. Gerö***

New Experiments on Interdiffusion of Electrolytes. pp. 45–55. ***T. Linhart***

Red Band Spectrum of Nickel Hydride. pp. 56–72. ***A. Heimer***

Rotation Structure of $^4\Sigma \rightarrow {}^4\Pi$ Bands. pp. 73–80. ***A. Budo***

Question of Origin of Earth's Magnetic and Electrical Fields. pp. 81–87. ***H. Haalck***

Experiments on Unipolar Electric Conductivity of Carborundum. pp. 88–91. ***M. Chakravarty and S. Khastgir***

Magnetostriction of Single Crystals at High Temperature. pp. 92–103. ***H. Takaki***

Range of Natural Protons. pp. 104–113. ***W. Volland***

Neutrino Theory of Light. II. pp. 114–121. ***P. Jordan***

Daily Curve of Lorentz Contraction from Observations of a Rotatable Level. pp. 122–126. ***L. Courvoisier***

Neutrino Theory of Light. pp. 127–132. ***M. Pryce***

VOL. 105, NO. 3–4, APRIL 14, 1937.

Nuclear Moment of Caesium from Inverse Zeeman Effect of Hyperfine Structure. pp. 133–163. ***T. Fölsche***

Refraction and Dispersion of Heavy Methane. pp. 164–167. ***T. Larsen***

Regularity in Nuclear Structure and Determination of Quadrupole Moment of $^{187}_{75}$Re and $^{185}_{75}$Re. pp. 168–174. ***H. Schüler and H. Korsching***

Near Infrared Absorption Measurements on Organic Dyes. pp. 175–192. ***A. Schiller***

Afterglow of Electron Beam Oscillograms with Screens Having Long Decay Times. pp. 193–201. ***M. v. Ardenne***

Symmetric Properties of Nuclear Forces, with Application to Nuclei with Up to 16 Particles. pp. 202–228. ***F. Hund***

Neutrino Theory of Light. III. pp. 229–231. ***P. Jordan***

Induced Radioactivity of Light Nuclei Produced by Alpha-Particles. pp. 232–250. ***A. Meye***

New Absorption Spectrum of Diatomic Sulphur. pp. 251–264. ***H. Cordes***